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SEARCH THEORY

by

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FORWORD

The research project, "Innovative Resource Planning in Urban Public Safety Systems," is a multidisciplinary activity, supported by the National Science Foundation, and involving faculty and students from the M.I.T. Schools of Engineering Science, Architecture and Urban Planning, and Management. The administrative home for the project is the M.I.T. Operations Research Center. The research focuses on three areas: 1) evaluation criteria, 2) analytical tools, and 3) impacts upon traditional methods, standards, roles, and operating procedures. The work reported in this document is associated primarily with category 2, in which a set of analytical and simulation models are developed that should be useful as planning, research, and management tools for planners and decision-makers in many agencies.

In this report Professor Morse provides a thorough tour of search theory for the planner who wishes to use and implement the results. The material covers approximately thirty years of development of the field, stemming from the original U.S. Navy Operations Research Group (1943), which was headed by Professor Morse. Although the vast majority of applications to date have been in the area of military operations, it is expected that more applications of search theory concepts will appear in an urban public safety setting. These could include, for instance, allocation of police preventive patrol or fire inspectors.

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6.0 Introduction

Being a part of a Handbook of Operations Research, this Chapter is addressed to the average worker in the field, not to the specialist in search theory. Results and conclusions are emphasized, rather than niceties of derivations (these can be found by going to the references). Procedural outlines and graphical aids are provided, so that use can be made of the theory in planning actual searches. The aim has been to foster such use, and the hope is that interest will be aroused in developing more usable solutions for real search problems.

Search is an example of an operations research subject wherein theory and practice have diverged as they have developed. Search theory, as a distinct subject of study, was begun in World War II* in response to a very practical need for the efficient use of planes and ships to find enemy submarines. The theory then worked out, rudimentary as it was, turned out to be of considerable help to the Navy in preparing search plans and procedures that were more effective than the earlier, more intuitive tactics. Since that time the mathematical logic underlying the theory has been appreciably strengthened (see,

*"Preliminary Report on the Submarine Search Problem," by P. M. Morse, R. F. Rinehart and others, issued in May 1942, was the first Technical Report of the newly formed Anti-Submarine Warfare Operations Research Group, financed by the National Defense Research Committee and assigned to the Office of the Chief of Naval Operations, Admiral King. It covered parts of the material in subsections 6.23, 6.32, and 6.52 below. The contents of this and of many other studies by various members of the Group are reported in consolidated form in Koopman (1946).

for example, Dobbie, 1968, and Pollock, 1971) and the range of suggested applications has been conjecturally extended, but it is questionable whether many of the later, more elegant, extensions are in a form to be of much help to an operator carrying out an actual search, with its attendant urgencies and errors.

Part of the reason for the lack of advance in applications is the wide variety of situations implied in the word search. Aside from the basic probabilistic principles, there is little in common between the computational search for the maxima of a complex function of many variables and the search for a lost child on the slope of a mountain or the search by the police for a fugitive who is continually changing his hiding place. If search theory is to extend its range of applicability, many specific practical cases will have to be analyzed in detail, and usable solutions (even though they be approximate and inelegant) must be found for each case.

As pointed out by Pollock (1971)(see also Danskin, 1962) the term search has at times come to encompass, not only the strategy of the operation of looking for a "lost" object or person (the target), but also the design and use of the detection equipment and the question of what to do after the object has been found. In this Chapter the less inflated definition will be accepted: search is the planning and carrying out of the process of looking for the target. We assume that the characteristics of the detection equipment have been obtained, either directly from operational experiments (as described in subsection 6.23) or else indirectly from the combined use of the statistical theory of signal detection and decision theory, and proceed to discuss how the equipment can be used in devising the strategy of actual search. A discussion of these excluded problems, and a partial list of related papers, is given in Pollock (1971).

This concentration on the actual search process allows us to shorten the list of appropriate measures of effectiveness.

In general we assume that the desideratum is to maximize the probability of finding the target, for a given expenditure of search effort. Occasionally we assume that the criterion is to minimize the expected time required to find the target; indeed in many cases these two criteria require the same strategies. We do not consider other criteria, such as maximizing the amount of information gathered, discussed by Mela (1961) and Dobbie (1968).

Even with this delimitation, the present Chapter, for reasons of space, can only include a review of basic principles, plus some scattered remarks regarding recent developments. Details of these recent developments may be obtained from the papers given in the bibliographies of Enslow (1966), Dobbie (1968) and Pollock (1971). Our discussion of basic principles will be given in terms of particular examples, to ensure that a modicum of realism be retained.

We will treat first the case of continuous search, because it has been studied in more detail. Here the military applications are more numerous, though other search situations have been dealt with. Some space is given to consideration of the effect, on the structure of an optimal search, of false targets that often dilute an actual search. Later sections deal with the problem of the search of discrete sites, with potential applications to the prospecting for ore or oil, or the police search for evidence. Finally the problem of the search for an active evader is touched on; here the theoretical development is just beginning and the application to practice is yet to come.

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6.1 Fundamental Concepts.

We start with the operation basic to nearly all physical search: that of a person searching with his eyes over an area, to see whether he can recognize some object or symbol or pattern (that we shall call the target) which he believes (or hopes) to be present somewhere in the area. Visual search displays nearly all the characteristics of more complicated search operations: the phenomenon of diminishing returns and the degree of improvement resulting from more orderly search patterns, for example. In addition, visual search is usually a component of more instrumented searches, in that the instruments -- the radar or sonar screen, for instance -- must be scanned visually.

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6.11 Visual Search of an Area.

The pertinent properties of the eyes in scanning an area and the nature of the psychophysiological response are reviewed in Chapter 4 of Koopman (1946). They form the basis of the following discussion.

The human eyes scan an area in a sequence of fixations in various directions (for about 1/2 to 1/4 second apiece) separated by rapid changes of eye direction; the eye does not "see" while the line of sight is moving. The detail seen per fixation drops off rapidly with angle away from the line of sight. Fine detail is perceived only by the fovea, the small central portion of the retina, subtending only a few degrees. Large objects, with a strong contrast, may be detected when 20° or more from the line of sight, but the central 2° to 5° are needed for fine detail.

These effects may be expressed in terms of a probability $\mu(\beta)$ that the searched-for target is recognized during a given fixation of the eyes in a direction at angle β to the line to the target. Probability μ of course depends on β, but also on the

(to page 4a)

illumination, the angular size and visual contrast of the target and of course on the state of the viewer's eyes. As noted, μ has a sharp maximum at $\beta = 0$, dropping rapidly to zero beyond $\beta \simeq 5^{\circ}$. The effective solid angle scanned for the target, per fixation, would then be the integral of μ over solid angle $d\Omega = d\alpha \sin\beta \, d\beta$,

$$\Gamma = \iint \mu(\beta)\, d\Omega$$

The value of Γ also depends on the nature of the target and on the conditions of illumination; it is a measure of the utility of a single glimpse in finding the target. In many cases this effective solid angle is small, of the order of 10 square degrees.

The magnitude of the search task also depends on the total solid angle Ω_s subtended, at the searcher's eyes, by the total area to be searched over. If initially the searcher has no idea of the position of the target in the solid angle Ω_s, and if the plane is oriented and illuminated so that Γ is independent of the direction to which the line of sight is pointed, then the chance of detecting the target in a single fixation, directed at random within Ω_s, will be constant, independent of direction (other cases will be considered later). This chance, called the a priori glimpse probability, the ratio between the effective solid angle scanned per fixation and the total solid angle to be scanned, $g = \Gamma/\Omega_s$, is the probability that the target will be recognized in a single, randomly pointed fixation.

When, as is usual in such visual searches, successive fixations are randomly directed, the probability that the object will be recognized in the n'th fixation is $(1-g)^{n-1}g$, the probability that it will still be undetected after the n'th glimpse is $(1-g)^n$ and the probability that it will be located by or before the n'th fixation is

$$P_n = 1 - (1-g)^n \qquad (1)$$

Since g is usually considerably smaller than unity and since searches of any importance involve hundreds of fixations (i.e., times of half a minute or more) this formula may be replaced by its asymptotic form

$$P_n = 1 - e^{-ng} \qquad (2)$$

It is often useful to express this formula in terms of time t spent and total solid angle Ω_s to be searched. If ν is the frequency of eye fixations during the search, so that $n = \nu t$, we can write

$$P(\Phi) = 1 - e^{-\Phi} \; ; \; \Phi = E/\Omega_s \; ; \; E = \omega t \qquad (3)$$

as the probability that the target will be detected in time t or sooner; where $\omega = \nu\Gamma$ is the <u>search rate</u>, in solid angle per unit time, E is the <u>total search effort</u>, in effective solid angle scanned in time t and Φ is the <u>specific search effort</u> or <u>sighting potential</u> of the search.

We note the important property of diminishing returns, paramount in all search procedures. Doubling the search effort E does not double the probability of finding the target. Other search operations, discussed later, correspond to less simple relations between $P(\Phi)$ and Φ, but for all well-organized searches the probability $P(\Phi)$ is related to the specific search effort Φ by the following general properties (see discussion of Eq. 21)

$$\left.\begin{array}{l} P(\Phi) \text{ is a monotonically \underline{increasing} function of } \Phi, \text{ and} \\ \quad P(0) = 0 \; ; \; P(\Phi) \to U \le 1 \text{ as } \Phi \to \infty \; ; \text{ furthermore} \\ P'(\Phi) \equiv (dP/d\Phi) \text{ is a monotonically \underline{decreasing} function} \\ \quad \text{of } \Phi \text{ and } P'(\Phi) \to 0 \text{ as } \Phi \to \infty \end{array}\right\} \qquad (4)$$

The adjective "well-organized" implies that the properties of $P(\Phi)$, expressed in (4), are in effect a definition of what we mean by "well-organized" search. At any time during the expenditure of search effort we should, if possible, direct our next quantum of effort in that direction that promises the greatest results; that is, for which P' is greatest then. If we can do this at every instant of the search, we will have picked first the action for which P' is the largest (or at least not smaller than for any other action), and so on; and P' will as a result be a monotonically decreasing function of Φ. For further discussion see Koopman 1956b, de Guenin 1961 and Section 6.33 of this Chapter). These properties of $P(\Phi)$, summed up in the phrase "diminishing returns", usually imply that the most efficient search involves a very non-linear distribution of search effort, as will be seen in Section 6.33.

6.12 <u>False Alarms, Non-random Scanning</u>.

Just now, however, we must return to an actual example of visual search, to see whether our assumption, inherent in Eq.(1), of the statistical independence of successive glimpse probabilities is (or can be made) valid, and whether the actual search rate ω is in practice equal to $\nu\Gamma$. For example, suppose a person is standing front of a large bookcase, trying to find a particular book that he believes is somewhere on the shelves. He first scans at random; then out of the corner of his eye he may glimpse what seems to be the right title and he directs his next fixations there to check. Perhaps the follow-up shows he was in error, so he returns to random scanning. Next time his attention is caught he may have to come close, or even to take the book off the shelf, before he realizes this also is not the book he is looking for. Eventually a glimpse, followed by a closer look, discovers the wanted book (if it is truly there).

Thus the actual process of visual search involves both random and correlated fixations. In addition, some of the "detections" prove to be false alarms, that tend to dilute the rate of search and thus delay the eventual discovery. In the latter part of subsection 6.41 we will discuss the effects of the presence of false targets on the structure of another kind of search, and in the latter part of subsection 6.42 we report the solution for a very simple false target situation (see Stone 1972 and Dobbie 1973 for further details). At present, however, the effect of false alarms on the visual search operation just described, has not been analyzed in detail, so the best we can do is to assume that it will not change the form of Eq.(3) but will reduce the magnitude of the search rate ω. In view of the number of approximations already imbedded in our assumptions, it is doubtful whether any more detailed analysis will produce a solution that is enough closer to what actually happens to warrant discarding the simplicity of Eq.(3).

In fact one can verify experimentally that the probability of finding a wanted book in a bookcase containing N books (N large) in time t is approximately given by the formula

$$P(\Phi) = 1 - e^{-\Phi} \quad ; \quad \Phi = \rho t/N \qquad (5)$$

where ρ, the effective search rate in books per unit time, depends on the searcher, the degree of illumination and the physical characteristics of the book, and includes the effects of false alarms. In practice this rate turns out to lie between 100 and 200 books per minute (see Morse, 1970) if the books are arranged at random on the shelves, so the a priori probability of the book's location is uniform throughout the bookcase. In this case the deviations to follow up false alarms slow the search but do not seem to alter its generally random nature.

The formula of Eq.(5) is an exemplar of the relationship between the probability $P(\Phi)$ of discovery and the specific search coverage $\Phi = E/A$, the ratio between search effort E and the area A to be covered. We note again the property of diminishing returns characteristic of all P's satisfying (4); if effort E is doubled, Φ is doubled but P is not doubled (unless E is small). Probability P is not additive, but search coverage is additive. For this reason Φ is often called the sighting potential (see Koopman, 1956b).

To measure the degree of inefficiency caused by this process of random fixation, we turn to the idealized situation of complete regularity of search. Suppose our eyes could be made to swing smoothly across area A and suppose the target would certainly be discovered if it came within a solid angle subtending a circular region R of diameter W on A, and would not be discovered if it were outside A (this assumption eliminates the effects of false targets). We could then try to cover area A efficiently by moving the line of sight so that region R sweeps out a regular, non-overlapping path, either in a spiral or a zigzag pattern, eventually covering all of A but never covering any area more than once.

If the target is equally likely to be anywhere in A, the probability that it will have been discovered by the time an area a of A had thus been searched over is

$$P(a) = \begin{cases} (a/A) & (a \leqslant A) \\ 1 & (a \geqslant A) \end{cases} \qquad (6)$$

where $a > A$ means that some of A has to be searched over again. To compare this with Eq.(5), for random search, we note that the sighting potential in the present case is $\Phi = a/A$. Thus the two curves for P start with the same initial value and slope at $\Phi = 0$ and both approach each other as $\Phi \to \infty$. The greatest difference between the two curves is at $\Phi = 1$, where the probability for uniform coverage is 1.00 and that for random coverage is 0.63. As we shall see later, the results for any intermediate degree of regularity in search coverage give results intermediate between these two curves (see Fig. 8).

6.2 Motion of the Searcher.

We turn now to a different operational situation, that of an aircraft flying over the ocean, searching for a ship or surfaced submarine. The plane's altitude is h; it is flying a straight course at speed v, which is great enough so the ship may be considered to be at rest. Here again we must discuss the rate of search, but in this case the rate is determined by the speed of the plane, which sweeps out a "searched strip" as it flies along.

6.21 Visual Search.

First consider the case of a visual observer, looking for the target ship. The most noticeable feature of a small ship, such as a submarine, is usually its wake, if it is moving. Thus, as he glances about, the observer's chance per glimpse of spotting it is roughly proportional to the solid angle the wake subtends at his eye, in addition to depending on the state of the sea and the transparency of the atmosphere. As indicated in Fig. 1, this solid angle is inversely proportional to $(r^2 + h^2)$ and proportional to $\cos\theta = h/(r^2 + h^2)^{\frac{1}{2}}$. In other words his glimpse probability for spotting the target when it is a horizontal distance r from the searching plane is $g(r) = Ch/(r^2 + h^2)^{\frac{3}{2}}$ where the value of the constant C depends on the size of the ship plus wake, its contrast to the surrounding sea (i.e., on the state of the sea) and on the range of atmospheric visibility. For further details see Koopman,1946

In a time dt, during which the plane will have moved a distance $dy = v\,dt$ in the y direction, the observer will have had time to make $\nu dt = (\nu/v)dy$ eye fixations. Following the discussion of the previous Section, the probability of not spotting the ship during time dt, when the ship is a horizontal distance r from the

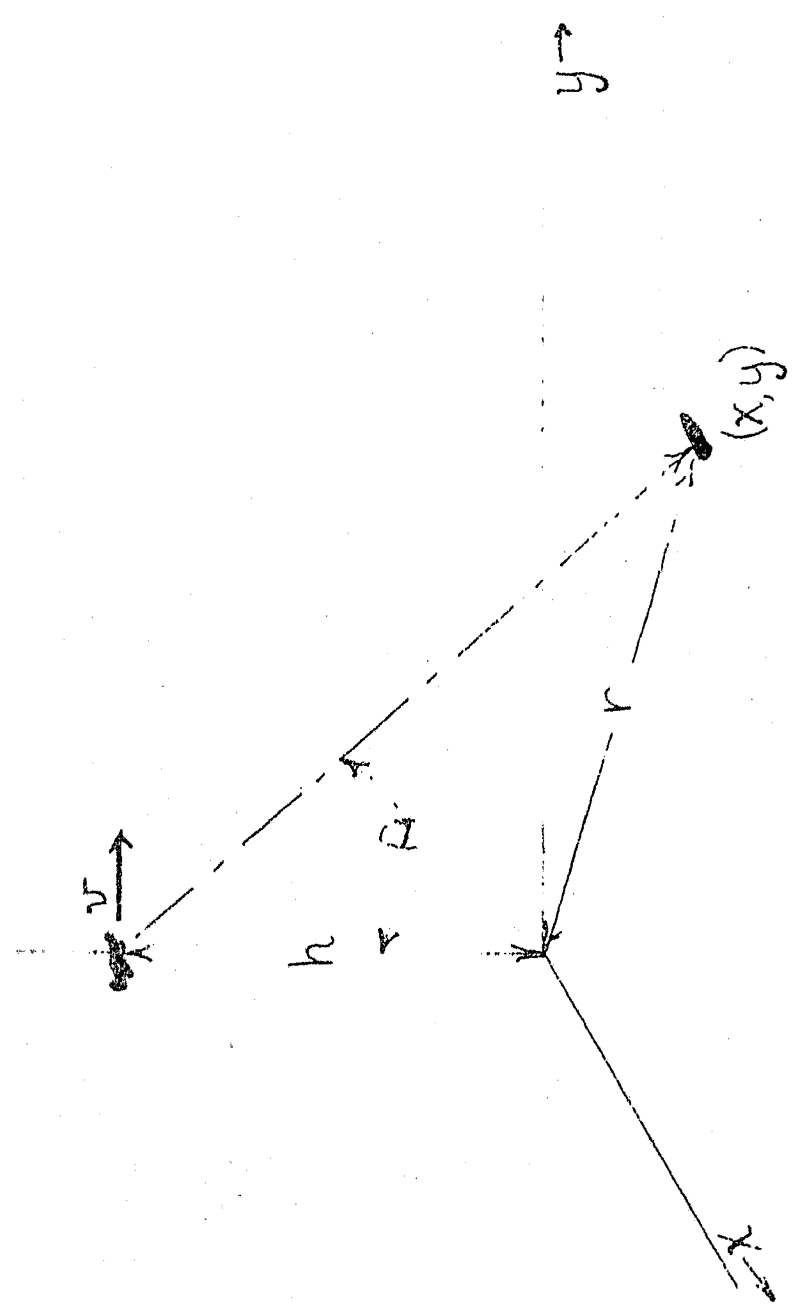


Fig. 1. Sighting of ship from a plane flying at altitude h, with relative velocity w in the y direction.

plane is $q(t) = \exp\left[-\nu Ch\,dt/(h^2+r^2)^{\frac{1}{2}}\right]$ and the cumulative probability of not finding the ship, as the plane progresses on its search course, is the product of all the partial probabilities, $\cdots q(t-dt)q(t)q(t+dt)q(t+2dt)\cdots$, for as long as the ship is within the solid angle searched over by the observer. Thus the probability $p = 1 - [\cdots q(t-dt)q(t)q(t+dt)\cdots]$ of finding the ship during the passage of the search plane is given by the equation

$$p = 1 - e^{-F(x)} \quad ; \quad F(x) = \int \nu g(r)\,dt \qquad (7)$$

where the integration is taken over the whole time during which the target is within the solid angle covered by the observer. The comments about visual search at the end of the previous Section indicate that the effective value of the constant νC is rather less than laboratory measurements would predict; indeed, to be safe, its value must be measured under operational conditions, as will be discussed later. Nevertheless, the general form of Eq.(7) is valid.

The quantity F(x) is, as mentioned previously, a sighting potential; its additive property is evidenced by its being an integral. If, later in the search, the plane's course brings it again within sighting range of the ship, the combined probability of detection would be obtained by adding the two values of F; $p = 1 - \exp(-F_1 - F_2)$. The individual F of Eq.(7) is a sum of all the infinitesimal sighting potentials accumulated as the plane passes by the target.

Returning to the formula for g(r) for visual sighting, we can work out the visual sighting potential for a ship that is a perpendicular distance x (called the lateral range) from the plane's course. It is

$$F(x) = \frac{kh}{v}\int_{y_0}^{\infty}\frac{dy}{(h^2+x^2+y^2)^{3/2}} = \frac{kh}{v(h^2+x^2)}\left[1 - \frac{y_0}{\sqrt{h^2+x^2+y_0^2}}\right] \qquad (8)$$

where y_0 is the rearward limit of the observer's scanned area, as shown in Fig. 2. If the observer scans the entire forward half of the ocean, $y_0 = 0$, the formula simplifies and the resulting probability of detection of a ship at lateral range x, from a plane at altitude h, travelling on a straight course with speed v, is

$$p(x) = 1 - \exp\left[-kh/v(h^2+x^2)\right] \simeq 1 - e^{-kh/vx^2} \quad (h \ll x) \qquad (9)$$

which is plotted as curve a in Fig 3.

In view of the discussion preceding Eq.(6), we see that the parameter k is likely to be rather smaller than νC. Nevertheless k is determined by the contrast and size of the sought object, the atmospheric visibility and the observer's alertness, plus the degree to which his position in the plane hinders clear vision in all directions (It may also depend on v if the plane is going very fast). The details of the methods of visual search also are important. For example, the use of binoculars may actually reduce the value of k, because such use reduces the frequency ν of fixations and also the size of the solid angle covered per fixation, even though it increases the probability of detection <u>if</u> the target is within the angle of view. Methods of measuring k under operational conditions will be discussed later.

6.22 Lateral Range Probabilities for Different Detection Devices.

The lateral range curve for visual search from a plane, curve a of Fig. 3, is one example of various curves corresponding to various instruments used to detect the searched-for target. A lateral range curve embodies the details of the search effectiveness of the detection equipment that is carried at uniform velocity v along a straight path that happens to pass a distance x

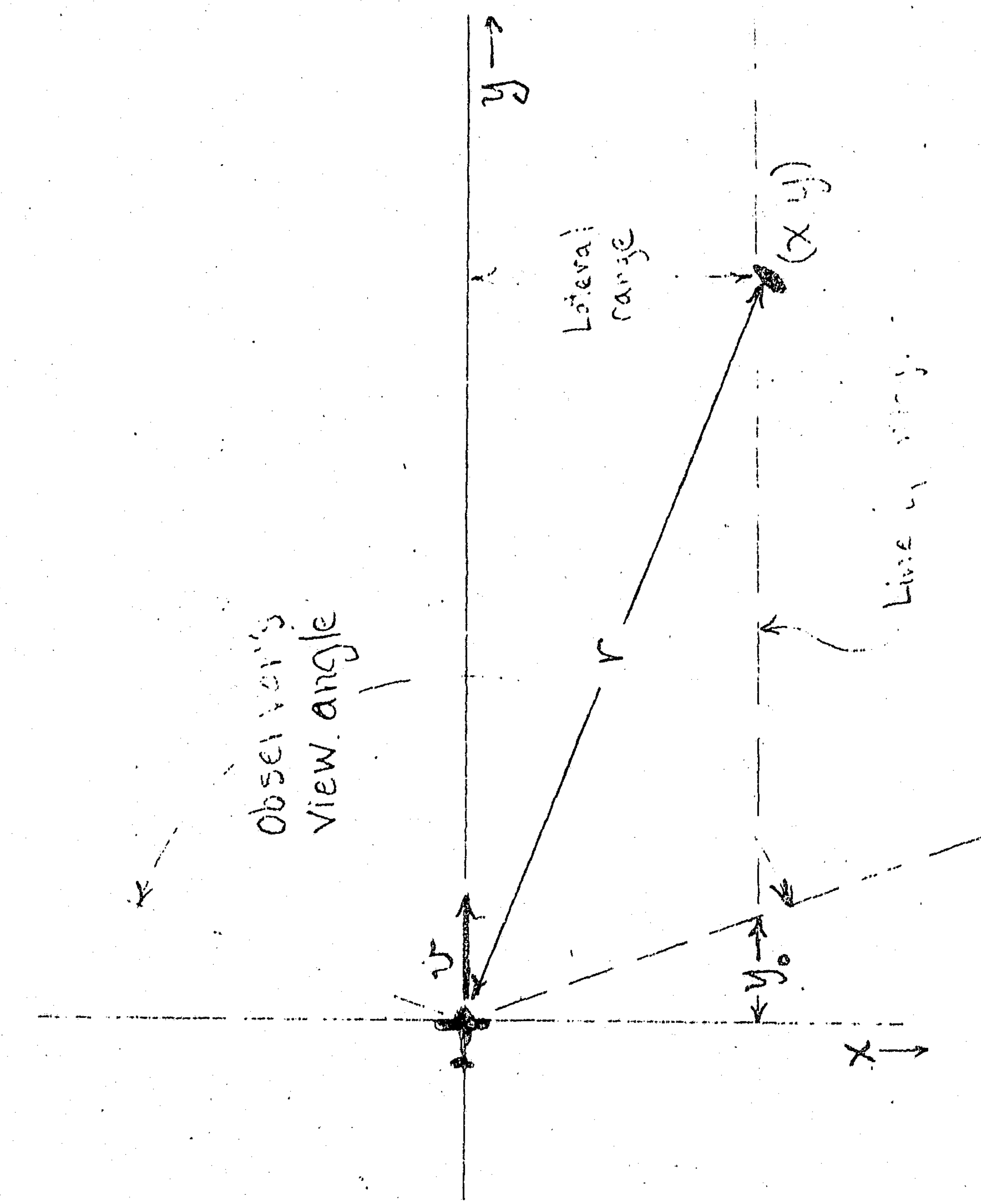


Fig. 2. Plan view of sighting.

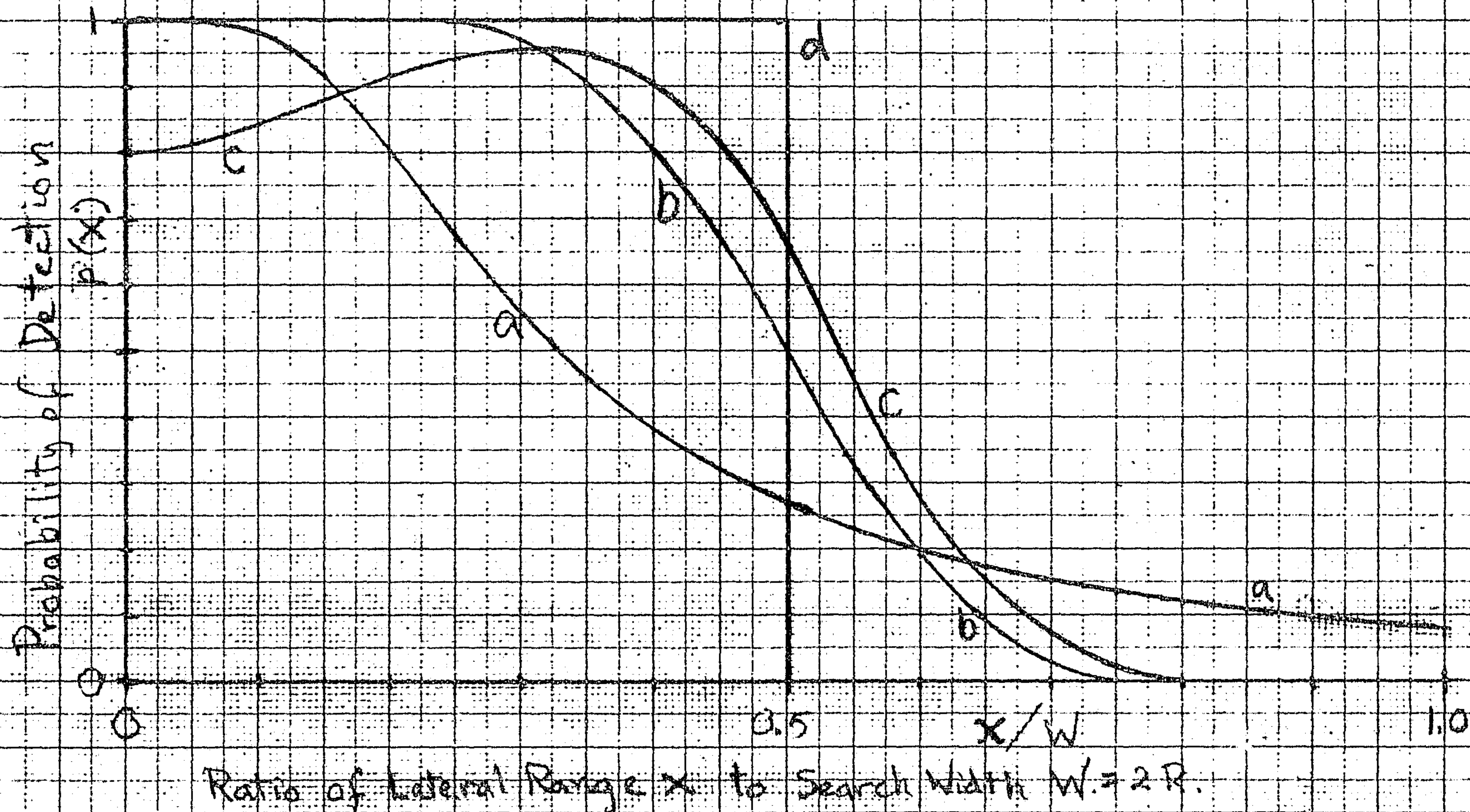


Fig. 3. Typical curves of probability of detection p(x), as function of lateral range x. Curve a is for visual search from a plane, b for radar, c for radar with sea return and curve d is for an idealized definite range law. All curves to scale W = 2R, as defined in Eq. 11.

from the target. The shape of the curve is dependent on the nature of the target and the type of detection equipment involved. In the idealized case where the object is not detected if it never comes within a definite range R of the observer, is certainly seen if it comes within range R, the curve is the <u>definite range curve</u> marked d in Fig.3.

In actual practice the lateral range curve seldom approaches the definite range curve d; nearly always there is a range of uncertainty near the limit of detection. For example, a search radar sends out a succession of pulses, as it swings its directional antenna around, and reflections from the target are received and displayed, as a "blip" on the scope, at a point corresponding to the position of the target with respect to the radar. If the terget is too small or too far away the received signal will be too small to produce a blip on the scope. If the signal is near the limit of detection a blip may occur only occasionally, instead of every time the antenna scans in the direction of the target. In addition, other objects, such as waves, produce blips that may intermittently show up on the screen. Only when the blip appears nearly every scan, i.e., only when the <u>blip-scan ratio</u> approaches unity, can the observer be sure that an object is really detected. (For further details, see Koopman, 1946, Chapter 5) Basically, radar search is a two-level search: the radar producing blips on the screen, the observer searching the screen for persistent blips.

There are analytic methods (see, for example, Pollock, 1971) to balance between the chance of false alarm and the chance of overlooking the target, in terms of the blip-scan ratio. Usually the exigencies of the search, and the stress on the searcher, preclude the application of such niceties in actual practice.

The degree of fatigue of the observer, for example, has been found have a much larger effect on the results than any prescribed rule for blip-scan ratio; observer fatigue can at times reduce the effective range of detection to half the optimal range. Also if plane speed is great enough, the observer may not have time to scan the scope thoroughly; P then will depend on v, as it does for visual search.

In any case, under reasonably good conditions, the lateral range curve for radar search would have the general shape shown in curve b of Fig.3. No detection occurs when the lateral range x is some factor (50% in the curve shown) greater than the effective range R; perfect detection occurs if x is less than R by about the same factor. Under poor conditions the curve may be more like curve c of Fig.3. For example the search plane may be flying over a rough sea, or over heavily wooded terrain, with a great number of false blips (sea or ground clutter) that tend to hide the true blip. In some cases the clutter is greatest in the forward direction, so the chance of detection is greatest for some intermediate value of x, as illustrated in curve c of Fig.3. For further details see Koopman, 1946, Chapter 5.

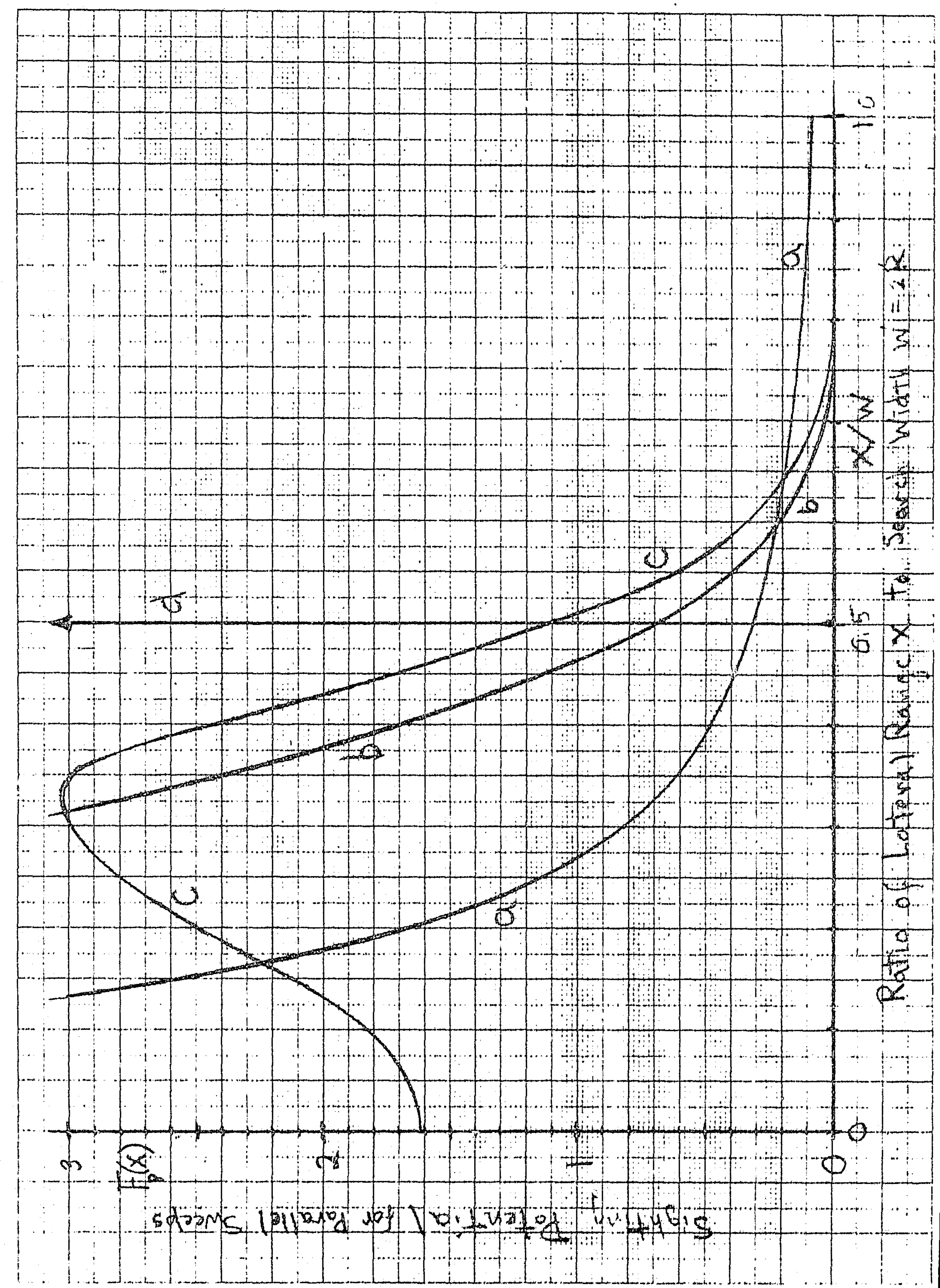
Most of the remarks made for radar search apply to the case of the use of sonar by a surface vessel searching for a submerged submarine (see Koopman, 1946, Chapter 6). Instead of ground clutter, the so-called reverberation tends to hide the true blip; also the signal tends to be lost when the vessel is nearly over the submarine. Therefore, except for the differences in distance scale, lateral range curves for sonar resemble curves b and c of Fig. 3.

Many other search situations correspond to the model discussed here. For example, the visual search, from a helicopter, for a lost child would probably conform roughly to curve a of Fig. 3, and thus to Eq.(9). If the person were lost in wooded territory, and the search has to be conducted on foot, the lateral range curve would more nearly correspond to b of Fig.3, the visibility being sharply limited by the trees. On the other hand, if shouts were used to alert the lost one, the shape may be nearer curve a. When dogs are used, still another curve may be appropriate. For further discussion of these problems, see Kelley, 1973.

The sighting potentials F(x), for the four curves of probability p(x), shown in Fig.3, are displayed in Fig.4. As mentioned before, these potentials are additive; if several observers are involved, either following along the same path or travelling in parallel paths, their potentials are to be added, to obtain the resultant probability of detection,

$$p(x) = 1 - \exp\left[-F_1(x) - F_2(x) - \cdots\right] \qquad (10)$$

For example, if the search plane has n visual observers, or if n planes follow the same path, the k of Eqs.(8) and (9) is to be replaced by nk. Because the probabilities follow the law of diminishing returns, such duplication of effort is inefficient unless F(x) for a single observer is less than about 0.7, or p(x) less than about 0.5. Thus additional sighting potential would be useful, in the visual case (curve a) for $|x| > 0.3\,W$.



6.23 Search Width and Its Measurement.

The effective width of the path swept out by the searcher in his course is found by integrating the probability of detection over the lateral range x,

$$W = \int_{-\infty}^{\infty} p(x)\, dx = \int_{-\infty}^{\infty} \{1 - \exp[-F(x)]\}\, dx \qquad (11)$$

where F(x) is the sighting potential. The search width W is the most useful single measure of the effectiveness of a detection instrument, carried by an observer moving in a continuous path over the area to be searched. As he moves, he can be reasonably certain to find the searched-for target if it comes within the swept path of width W, centered on his track. As one can see from Fig.3, most search will not certainly detect the target if it lies between $\frac{1}{2}W$ and $-\frac{1}{2}W$ of the path; but there is a compensating chance of finding it if it lies beyond $\pm\frac{1}{2}W$, so the effective width is W.

The effective search width for visual search from low altitude is, according to Eq.(9),

$$W \simeq \int_{-\infty}^{\infty} [1 - \exp(-kh/vx^2)]\, dx$$

$$= 2\left[x(1 - e^{-kh/vx^2})\right]_0^{\infty} + 2\int_0^{\infty} (2kh/vx^2)e^{-kh/vx^2}\, dx$$

$$= 2\sqrt{\pi kh/v} \qquad (\text{for } h < W/10) \qquad (12)$$

(Note the difference with Eq.29 of Koopman, 1956,2; we assume the observer looks in the forward half circle, instead of all around). For higher altitudes, an approximate formula is

$$W \simeq 2\sqrt{\pi kh/v}\, \exp(-hv/4k) \qquad (13)$$

We note that the width increases with altitude up to $h \simeq (2k/v)$, above which visibility begins to reduce the chance of spotting the target, even when below the plane. We note also that increasing

the speed of the search plane reduces the search width, which is not surprising, since increasing the speed of the plane shortens the time during which any given area is scanned. As was noted before, if n independent observers traverse the same track, the k in the square root is replaced by nk.

Each of the curves of Fig.3 and Fig.4 have the abscissa scaled to the effective search width. The curve for the usual radar and sonar search (b in Figs.3 and 4) is much closer to the idealized definite range curve d, than is the visual search curve a; the fringe of low probability for curve b does not extend very far beyond $x = W/2$ and over the range $0 < x < W/2$ the chance of detecting the target is nearly unity. In this case it would be a nearly complete waste of effort for another radar or sonar vehicle to repeat the same path (unless the seeing is poor, as with curve c).

We have noted earlier that the ability of an observer, with his vehicle and equipment, to detect some target, depends on so many variables that in practice it is wellnigh impossible to predict this ability from laboratory measurements. Thus, if it is important to conserve search effort (and, for this, one needs to know the value of W), the only safe procedure is to measure W under conditions closely approximating those in actual search. It was found, in World War II, that the usual search width W for radar planes searching for German submarines, was one half to one third the value claimed by the radar manufacturer, based on laboratory measurements. This is not surprising when one compares the results of tests on an optimally tuned radar, operated by an expert, with the results using a radar that had seen heavy service, operated by a tired G.I. If the manufacturer's claims had been used in planning, there would have been large "holes" in the search plans. In addition to these differences between laboratory and practice, there is the effect of "false alarms" and the pauses to verify questionable detections (discussed in Sections 6.1 and 6.4) which serve to dilute the search effort by amounts that can usually only be determined experimentally.

If a target simulating the real target is easy to construct, the measurement can be carried out as follows. Lay out a band of width D, at least 3 times the best estimate of the search width W and of length L at least 10 times D, with a well-marked, straight search course down its middle. Now place T simulated targets, more or less uniformly distributed over the whole area LD, but not so regularly spaced that it would be possible to deduce the regularity. If one wishes to measure p(x) as well as W, the distance from the search path of each target should be measured, and recorded. An observer is then sent along the search path and required to note the position of each target he observes during his passage. After checking his records and removing the "false alarms", if it turns out he has spotted n of the T targets then an estimate of the search width is nD/T. If n is larger than $\frac{1}{2}$T, the band width D was chosen too small, D should be doubled, the targets redistributed uniformly over the new band, and the experiment run over again.

If the number n of true targets spotted is less than about 20, statistical fluctuations will preclude accuracy in the result. In this case a number m of independent observers should be run through the course, making sure that each is ignorant of the location of the targets or of the findings of other observers. When the total number $N = \sum n$ of targets spotted by all m observers reaches a value of 100 or more, the resulting ratio (ND/mT) will be a

reasonably accurate estimate of the search width W.

If one can persist long enough for N to reach values of 500 to 1000, and if the lateral range of each target has been measured, then a rough estimate of the lateral range curve of Fig.3 can be constructed. One divides the N spotted targets (counting each target as many times as it has been spotted, as before) into those within W/6 on either side of the search path (suppose there are N_1 of these), those with lateral range between $\pm$(W/6) and $\pm$(W/3) (N_2 of those), those with lateral range between $\pm$(W/3) and $\pm$(W/2) (N_3 of these) and so on until all the N have been counted. One can then construct a block diagram, as shown in Fig.5, with the height of the i'th block equal to $(3N_i/N)$, which will be a rough estimate of the lateral range curve, as shown in Fig.5 by the solid line. The accuracy of the result depends, in part, on the uniform distribution of the initial placing of targets; there should be a roughly equal number in each of the strips parallel to the path.

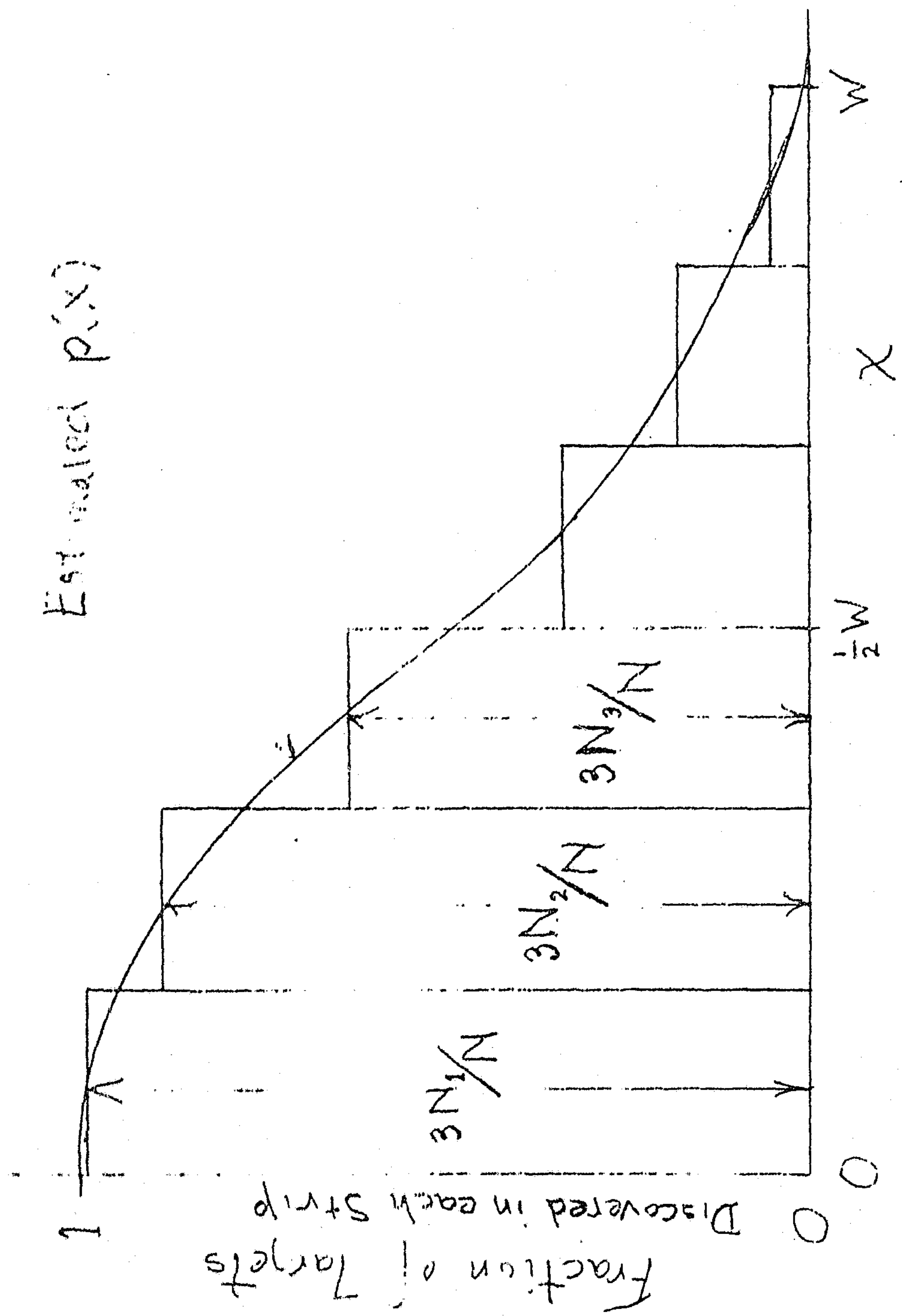


Fig.5. Block diagram, illustrating experimental determination of lateral range curve, p(x).

6.3 Search of an Area.

The usual search operation involves the covering of an area to find some target presumed present. In this Section we assume that there is no initial guess as to the object's whereabouts, so one has to assume it is equally likely to be anywhere in the area. In accord with the discussion following Eq.(5), and dealt with further in Section 6.33, the best way of applying our search effort, in this case, is to distribute it as evenly over the whole area as is operationally possible. Details of the derivation of many of the equations given in this Section may be found in Koopman, 1956,b .

6.31 Parallel Sweeps.

If the area to be searched is considerably larger than can be covered by a stationary inspection or by a single sweep through it, the best way of insuring uniform coverage is by a sequence of parallel sweeps, spaced a distance S apart. This may be accomplished by a spiral path or by the zigzag course of a single observer, as shown in Fig.6, or else by a number of observers following parallel courses interspaced a distance S. Depending on the time and degree of effort available, n parallel courses can be afforded, each of length D, spaced S = C/n apart, thus amounting to a total path length L = nD. From the previous Section we have measured an effective search width W, so that WL = nDW = A(W/S) is the area effectively searched (or the <u>total search effort</u>) A = CD being the area to be searched and WL/A = W/S being the <u>fractional search coverage</u> (or <u>specific search effort</u>, or <u>total sighting potential</u>). If the dimensions C and D of the area are considerably larger than W, n and thus L can be considered to be

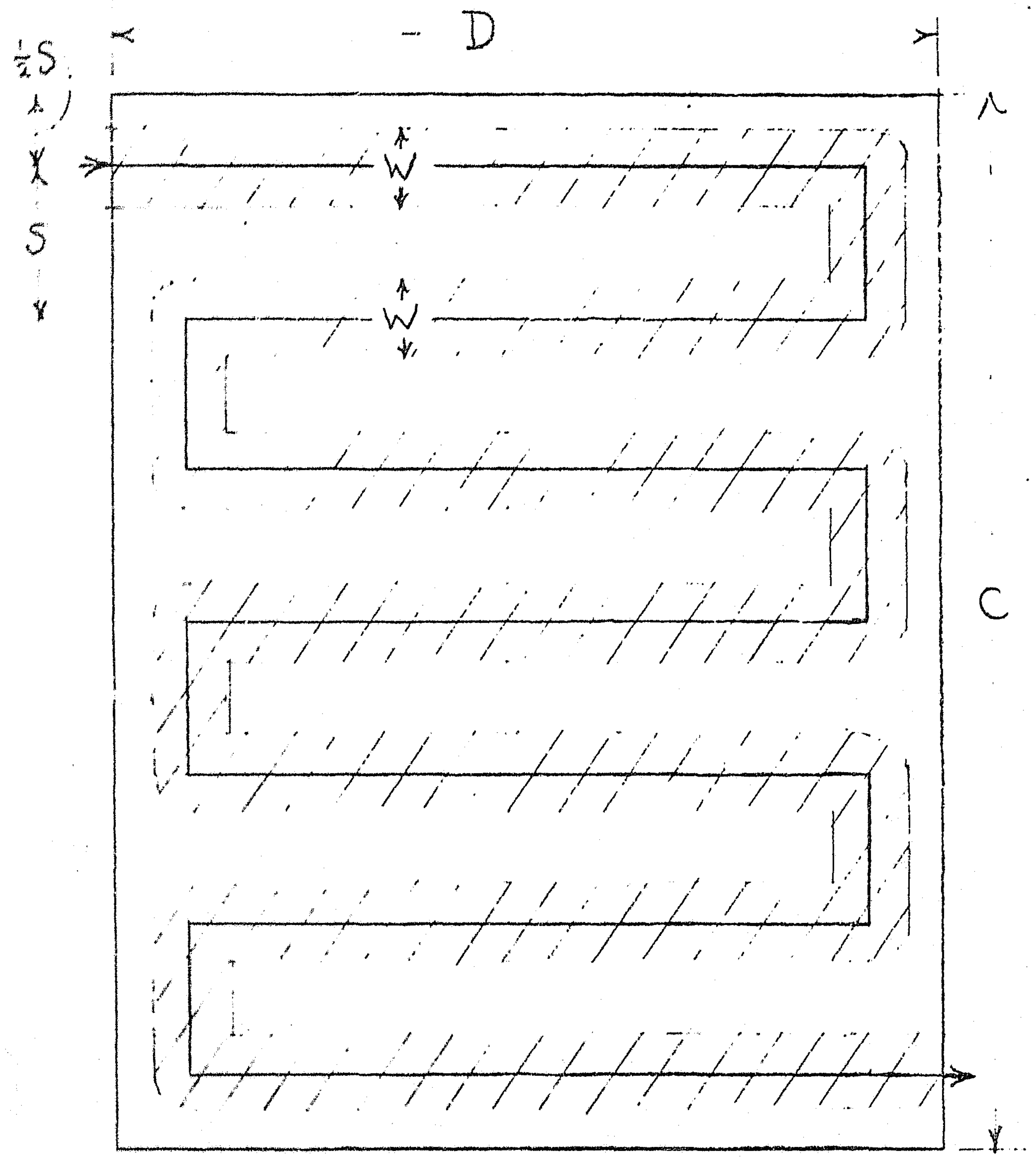


Fig.6. Parallel sweeps, of search width W, spaced a distance S apart, providing uniform coverage of area A = CD.

continuous variables. Also, if the area A is not rectangular, as shown in Fig. 6, but is sufficiently large and compact in shape, a spiral pattern or one of parallel sweeps can be laid out that produce essentially the same uniform coverage and the same fractional coverage for the same total search effort WL.

To predict the probability of finding the target during such a coverage, we must add the sighting potentials F of Fig.4, for the parallel sweeps as shown in Fig.7. We have (if A is large enough)

$$p_p(x) = 1 - \exp[-F_p(x)]$$
$$F_p(x) = \sum_{n=-n_o}^{n_o+1} F(|x - nS|) \quad ; \quad \text{see Fig.7} \qquad (14)$$

for the probability of detection if the target has a lateral range x from one of the paths. The formulas represent the fact that each parallel sweep contributes its share to the total sighting potential F_p. The limiting value n_o is the integer such that F(x) becomes negligible for $|x|$ between n_oS and $(n_o+1)S$; usually F is such that n_o is small. Thus in practice we need not consider "edge effects" for the sweeps next to the edges of the area. If these edge effects are neglected, both F_p and p_p are periodic functions of x with period S.

Let the distance of the searched-for target from edge D be z. of the target is equally likely to be anywhere in A, it is equally likely for z to have any value between 0 and nS = C. Thus the probability that the object will be found by the end of the n sweeps is an integral of the periodic function p_p over the whole width C, divided by C,

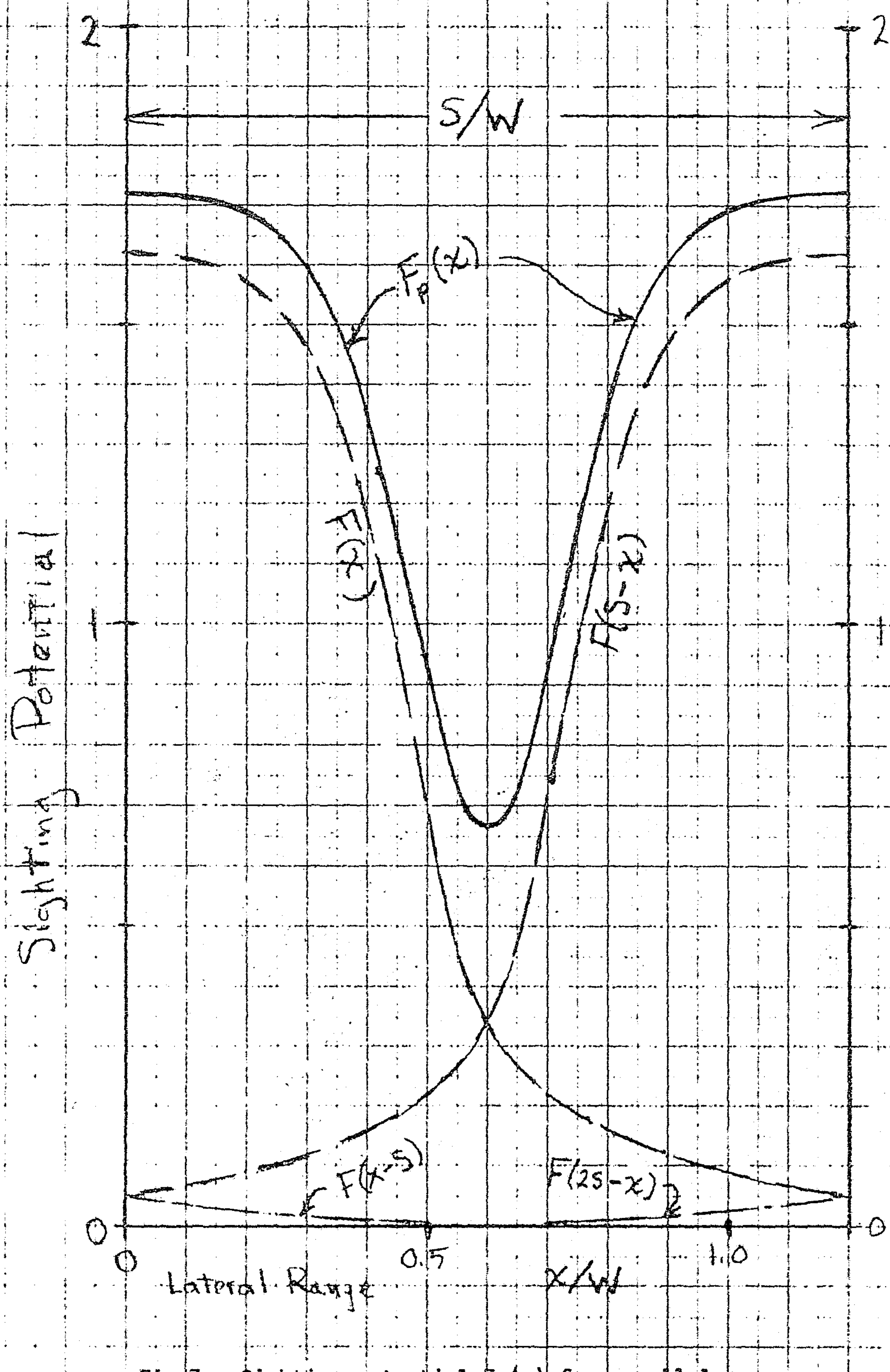


Fig. 7. Sighting potential $F_p(x)$ for parallel sweeps a distance S apart.

$$P(\Phi) = \frac{1}{nS}\int_0^{nS} p_p(z - \tfrac{1}{2}S)\,dz = \frac{1}{nS}\int_{-\frac{1}{2}S}^{(n-\frac{1}{2})S} p_p(x)\,dx = \frac{1}{S}\int_0^S p_p(x)\,dx$$
$$= \frac{1}{S}\int_0^S \left\{1 - \exp\left[-F_p(x)\right]\right\} dx \quad ; \quad \Phi = \frac{W}{S} = \frac{WL}{A} \qquad (15)$$

The integration can be carried out analytically for the visual search case of Eq.(8). The result (when h is small enough so that $F \simeq kh/vx^2$) is

$$P(\Phi) = \mathrm{erf}\left(\frac{\pi}{S}\sqrt{\frac{kh}{v}}\right) = \mathrm{erf}\left(\tfrac{1}{2}\sqrt{\pi}\,\Phi\right) \qquad (16)$$

where $\mathrm{erf}(u) = (2/\sqrt{\pi})\int_0^u e^{-t^2}\,dt$ is the well-known error function. Details of the calculation may be found in Koopman, 1956,b (note the misprint in his Eq.45). Other cases may be calculated numerically, once the appropriate sighting potential F(x) has been determined by measuring p(x) operationally and computing $F(x) = \ln\left[1 - p(x)\right]$, then calculating F_p and p_p and finally integrating p_p/S numerically from $x = 0$ to $x = S$.

The curves for $P(\Phi)$, as functions of the fractional search coverage $\Phi = WL/A$, are shown in Fig.8, for parallel sweeps of equipment exhibiting the different detection capabilities displayed in the lateral range curves of Fig.3. All of them show a decided dimunition of value when $\Phi = W/S = WL/A$ becomes smaller than unity, but a substantial diminishment of additional returns when Φ becomes larger than unity. Also the curves a, b and c are not greatly different from the limiting curve d, for the definite range law. The implications of these properties, when the *a priori* probability of presence of the target is *not* uniform throughout A, will be discussed in the next section.

Curve b, for radar search under good conditions, is nearly identical with the definite range curve d. In both cases an increase of specific search effort Φ greater than unity produces practically no additional sightings. (If, however, too optimistic

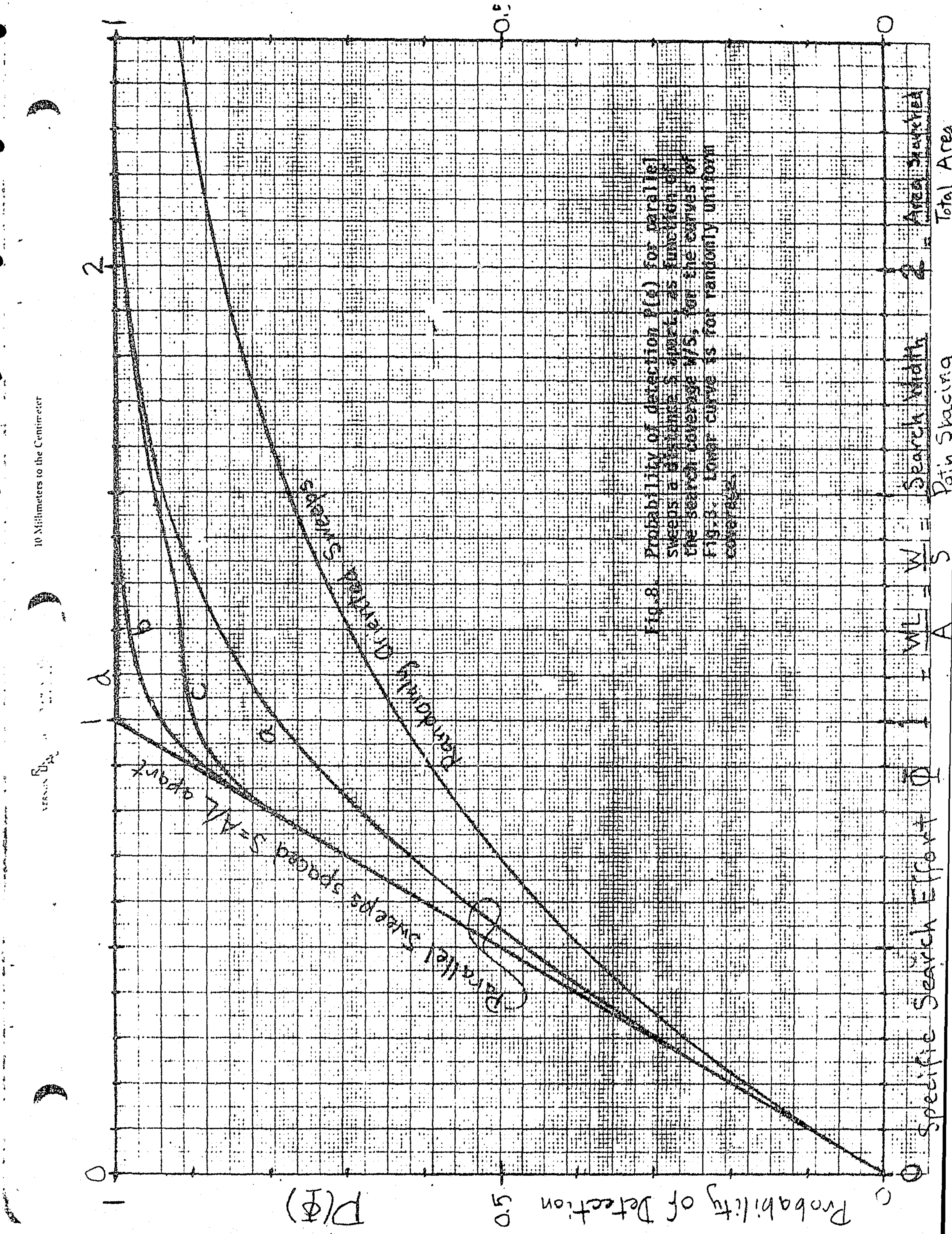


Fig.8. Probability of detection P(Φ) for parallel sweeps a distance S apart, as function of the search coverage W/S, for the curves of Fig.3. Lower curve is for randomly uniform coverage.

a value of W is used, one may think $W/S > 1$ and no further effort is needed, whereas in actuality $W/S < 1$ and more search could profitably be applied).

Curve c, for radar under difficult conditions of sea or ground return is an interesting case because of the peaks in $F(x)$ and $p(x)$ for $|x| > 0$ (see Figs.3 and 4). This results in a curve for $P(\mathbb{W})$ that nearly flattens out when the peaks of $F(x)$ and $F(S-x)$ coincide, and then rises again slowly as S is further decreased (or $\mathbb{W} = W/S$ is further increased). Thus $P'(\mathbb{W}) = dP/d\mathbb{W}$ has a minimum value and then rises again to a subsidiary maximum as $\mathbb{W}$ is increased further, before dropping asymptotically to zero. Thus this curve does not meet the requirements of (4) for a well-organized search. As will be indicated later, this makes it difficult to optimize the search effort. Luckily the conditions giving rise to curve c do not arise in practice very often.

6.32 <u>Randomly</u> <u>Distributed</u> <u>Sweeps</u>.

In actual searches it often is quite difficult to traverse the precisely parallel, equally-spaced tracks assumed in the previous subsection. In fact, unless all the tracks are laid out <u>and checked</u> during execution by accurate visual or radar triangulation, it is unlikely that the optimistic calculations of detection probability, indicated in curves a to d of Fig.8, can be achieved. It is much more likely that the results will correspond more nearly to an assumption that the path or paths cover area A more or less uniformly but are randomly oriented. To be more precise the likelier model is that of search paths made up of a number of straight segments (as sketched in Fig.9a) of total

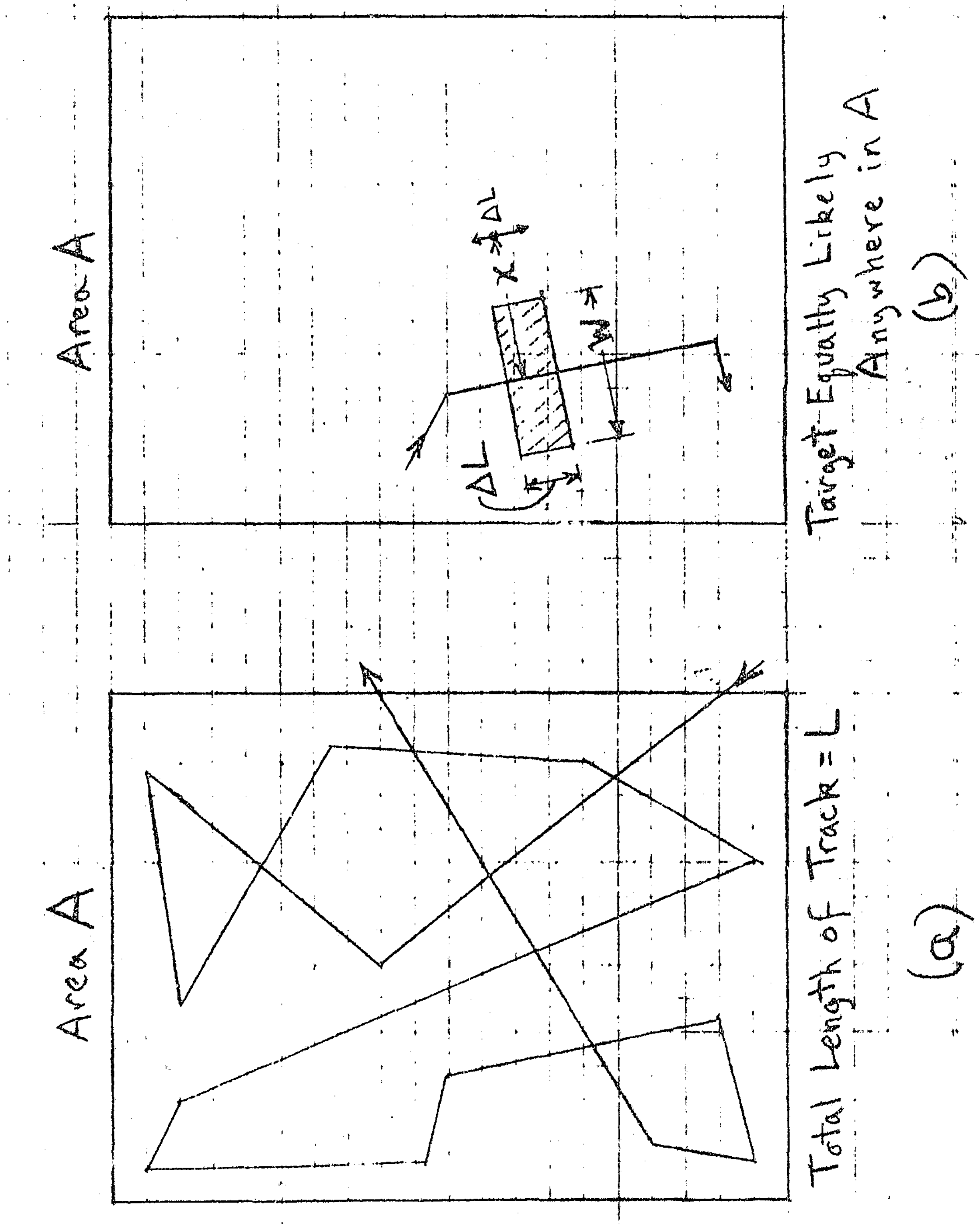


Fig.9. Randomly oriented sweeps to cover A more or less randomly.

length L within A, with the position and orientation of one segment being independent of the position and orientation of any other segment that is separated from the first by several intermediate pieces.

To analyze this case we examine the length ΔL of search track, as shown in Fig. 9b. If the target is equally likely to be anywhere in A, and if the search track is randomly located, in the sense of the previous paragraph, then the target is equally likely to be anywhere in relation to the element of track ΔL. If the detection equipment has a definite range $\frac{1}{2}W$, as indicated by curve d of Fig. 3, the chance of detection, while traversing ΔL is the area of the shaded rectangle of Fig. 9b, $W\Delta L$, divided by A. The result is the same for any lateral range curve, for the probability the target is in the elementary area $dx\Delta L$, a distance x from the track, is $dx\Delta L/A$ and thus the chance of detection during ΔL by equipment having the lateral range probability p(x) is

$$P = (\Delta L/A)\int_{-\infty}^{\infty} p(x)\,dx = (W\Delta L/A) \qquad (17)$$

according to the definition of search width W, given in Eq.(11).

The argument now proceeds as did that of Section 6.1, and reaches a similar result. The chance of not finding the target while traversing the element ΔL is $[1-(W\Delta L/A)]$ and the chance of not finding it in a sequence of n elements is $[1-(W\Delta L/A)]^n$. Since there are $n = L/\Delta L$ such elements in the total track traversed in the search, the chance of finding the target during the search is

$$P(\Phi) = 1 - \left[1 - \frac{W\Delta L}{A}\right]^{L/\Delta L} \xrightarrow[\Delta L \to 0]{} 1 - e^{-\Phi} \qquad (18)$$

where $\Phi = (WL/A)$ is (as previously) the effective search potential or specific search effort. The curve for this probability is shown in Fig.8, along with the curves for the parallel sweeps of the previous subsection.

Thus, as soon as random deflections disorganize to any extent a parallel search pattern, no matter what the detection equipment, the probability of success reduces to the same exponential dependence on search effort as was found in Section 6.1 (Eq.5) for the simplest sort of visual search. Of course the constants involved, expressed in terms of W and L, differ in value from the ω and t of Eq.(3), depending on the nature of the detection equipment and its carrier. Nevertheless the similarity in form of the resulting equation for $P(\Phi)$ means that we can develop procedures for optimal allocation of search effort that are almost completely independent of the nature of the search, as long as it involves covering an area.

The preceding discussion, however, should not be used as an excuse to be careless in laying out and following a search track. A glance at Fig.8 shows that the detection probability for random sweeps is less than any of the probabilities for parallel sweeps, for the same amount of effort. Parallel sweeps should be used whenever possible, but one should be sure that the paths are accurately parallel and equally spaced, or one runs the danger of overestimating the search effectiveness. Finally, it should be realized that to cover an area uniformly, even with randomly oriented sweeps, requires a fair amount of planning and path control.

6.33 Miscellaneous Examples.

Isbell (1957) and Gluss (1961) have discussed a very different "search" problem, where the target is very large, the searcher is blind and must find the target by moving around until he bumps into it. For example the "lost at sea" problem assumes that one is in a dense fog, that he knows exactly how far from shore he is but has no idea in what direction it is. The problem is to devise a path that either minimizes the maximum distance travelled (Isbell, 1957), or minimizes the statistical expectation of the distance to be travelled (Gluss, 1961).

Search for other large objects have also received attention. For example Gluss (1961b) has worked out the optimal path for finding (by touching) a circle of known radius and distance away, but unknown direction. The result could also be useful in trying to find a point target a known distance L away, direction unknown, by use of detection equipment with known definite range R for the target (L>R). However these exercises are of limited utility in practice because of the assumption of precise knowledge of target distance, and the solutions are quite sensitive to these assumptions.

Other problems, discussed by Bellman (1962) and, for example, Heyman (1968), involve the "search" for maxima (or zeros) of a function of many variables by means of dynamic (or linear) programming. A survey of this work would lead us too far afield from the practical problems surveyed in this Chapter.

6.4 Optimal Allocation of Search Effort.

If nothing is known as to the position of the target, except that it is inside area A, the optimal procedure is to distribute the available search effort uniformly over A. This is easily verified if one considers the search coverage $E_j = WL_j$ to be a function of each subarea A_j in A. The probability of finding the target in A_j is the product of the probability A_j/A that the target is in A_j, times the probability $P(\phi_j)$, $(\phi_j = E_j/A_j)$, that it would be found *if* it were in A_j, so the total probability of success is $P(\phi) = \sum(A_j/A)P(\phi_j)$. If the coverage is made non-uniform by making ϕ_j somewhat greater (by an amount $\Delta\phi$, say) than the average potential $\phi = WL/A$ and, in consequence, making E_i/A_i for another equal subarea $A_i = A_j$ less by the same amount $\Delta\phi$ then, because $P(\phi)$ is subject to the law of diminishing returns, the total probability $P(\phi)$ will be reduced for all the cases so far discussed. Since $P(\phi+\Delta\phi) - P(\phi) < P(\phi) - P(\phi-\Delta\phi)$, the total probability will be diminished by the negative amount $(A_i/A)\left[P(\phi+\Delta\phi) + P(\phi-\Delta\phi) - 2P(\phi)\right]$.

6.41 The Effect of Some Knowledge of the Target's Whereabouts.

Now suppose something *is* known about the target's location, so that its *a priori* probability of presence varies from region to region within A. We first deal with the rather impractical general case, when the *a priori* probability density $g(r)$, of its being at the point indicated by the vector r, is known to vary from point to point within A. Since $g(r)$ is a probability density

$$\iint_A g(r)\,dA = 1 \qquad (19)$$

If the density of search coverage $\phi(r) = \lim_{A_j \to 0}(WL_j/A_j) = d\Phi/dA$ at point r also may differ from point to point in A, the probability $\mathcal{P}(\Phi)$ of detecting the target, during the expenditure of a given search effort

$$\Phi A = \iint_A \phi(r)\, dA = WL \qquad (20)$$

throughout A, is the integral of the product of the probability g(r)dA of target presence in dA and the probability $P[\phi(r)]$ that the target is found if it is in dA,

$$\mathcal{P}(\Phi) = \iint_A g(r)\, P[\phi(r)]\, dA \qquad (21)$$

Suppose two distributions of search density (each adding up to the same total coverage Φ) are compared; one being $\phi(r)$ and the other $\phi(r) + \delta\phi(r)$, differing from ϕ by the relatively small amount $\delta\phi$ (such that $\iint \delta\phi\, dA = 0$) (so that the integral of both ϕ's over A equals the same ΦA). The difference in total probability of detection $\mathcal{P}$ will be

$$\Delta\mathcal{P} = \iint_A g(r)\left[P(\phi) + \delta\phi P'(\phi) - P(\phi)\right] dA$$
$$= \iint_A \delta\phi\, g(r)\, P'[\phi(r)]\, dA \qquad (22)$$

where $P'(\phi) = dP/d\phi$. The distribution of search effort $\phi(r)$ will yield the maximum probability of detection $\mathcal{P}(\Phi)$ when $\Delta\mathcal{P} = 0$. And the only way $\Delta\mathcal{P}$ can be zero, for <u>any</u> choice of $\delta\phi$ (as long as $\delta\phi$ is small and $\iint \delta\phi\, dA = 0$) is for the product $g(r)P'[\phi(r)]$ to be a constant, G, independent of r.

Of course this optimal distribution ϕ must satisfy Eq.(20), that the integral of ϕ over A must equal the specified total search effort $WL = \Phi A$. The requirement, arrived at in the last paragraph, that $P'(r) = G/g(r)$ if $\mathcal{P}$ is to be maximum, may involve an inconsistency; for the curves of Fig.8 show that the maximum value of $P'(\phi)$ is unity (when $\phi = 0$), no matter which curve is used. Now if, for any value of r, the <u>a priori</u> probability

density of presence of the target is less than the value of the constant G, the requirement that $P' = G/g$ cannot be satisfied, so this region must have zero coverage. No region in A, for which $G/g > 1$ can be covered by the search, if it is to be optimal; any effort would more effectively be used in an area where $G/g < 1$. Thus the properties of the search operation outlined in (5), prescribe a highly discriminatory search plan, if search effort is not limitless. Details of the derivation of this formula, and proof that the resulting $\mathcal{P}$ is a maximum, not minimum, are given in de Guenin, 1961 and Dobbie, 1963.

More explicitly, the procedure for computing the search coverage $\phi(r)$ that maximizes the probability $\mathcal{P}(\Phi)$ of detection of the target for a specified total search effort $\Phi A = WL$, is a two-phase one:

1. The appropriate value of the constant $G = g(r)P'[\phi(r)]$ may exclude some portions of area A, those for which $G/g(r) > 1$. In this excluded area A_e the search density is to be zero. In the searched area $A_s = A - A_e$ the density of search $\phi(r)$ must be such that $P'[\phi(r)] = G/g(r)$, which, in A_s is everywhere less than or equal to unity.

2. In addition, G must satisfy the requirement that the integral of the search density ϕ, as specified in 1, over the searched area A_s, be equal to the specified search effort $\Phi A = WL$.

These requirements can be more compactly stated in terms of the inverse function of $1/P'(\phi) = g/G$. Call it $f(g/G) = \phi$, so that $1/P'[f(g/G)] = g/G$, and $f[1/P'(\phi)] = \phi$. Then, to maximize the probability of detection in an area A, within which the *a priori* probability density of presence of the target at r is g(r), for a given total search effort, defined as $WL = \Phi A$, we find a value of G such that

$\iint_{A_s} f[g(r)/G]\, dA = \Phi A$, with the integration over the portion of area A_s within which $g(r)/G \geq 1$. Then the optimal search density at r is $f[g(r)/G]$ in A_s and zero in $A_e = A - A_s$, where $g/G < 1$. The resulting maximal probability of detection is then

$$\mathcal{P}(\Phi) = \iint_{A_s} g(r)\, P\{f[g(r)/G]\}\, dA \qquad (23)$$

Curves of $f(1/P')$, for the cases shown in Fig.8, are plotted in Fig.10.

One limitation of this procedure comes from the assumption that $P'(\phi)$ has a single-valued inverse function f. This is the case for curves a, b, and d and the random coverage curve of Fig.8. However curve c does not have a single valued inverse because its P' is not a monotonically decreasing function of ϕ. As long as g/G does not rise above about 15 over the whole of A, we can ignore the complication, but if the available search effort is large enough so that $g/G > 20$ over some portion of the area then a part of the effort must be dense enough to make up for the "valley" at $x = 0$ in the lateral range curve. Because this type of curve is rarely encountered in practice, we shall devote no further space to its idiosyncrasies.

Even with "normal" inverse functions $\phi = f(1/P')$, which are single-valued functions of $1/P'$, the features of procedure (23) do not correspond to intuitive allocations of search effort. The fact that regions of low *a priori* probability of presence should be avoided entirely is due to the fact that the maximum

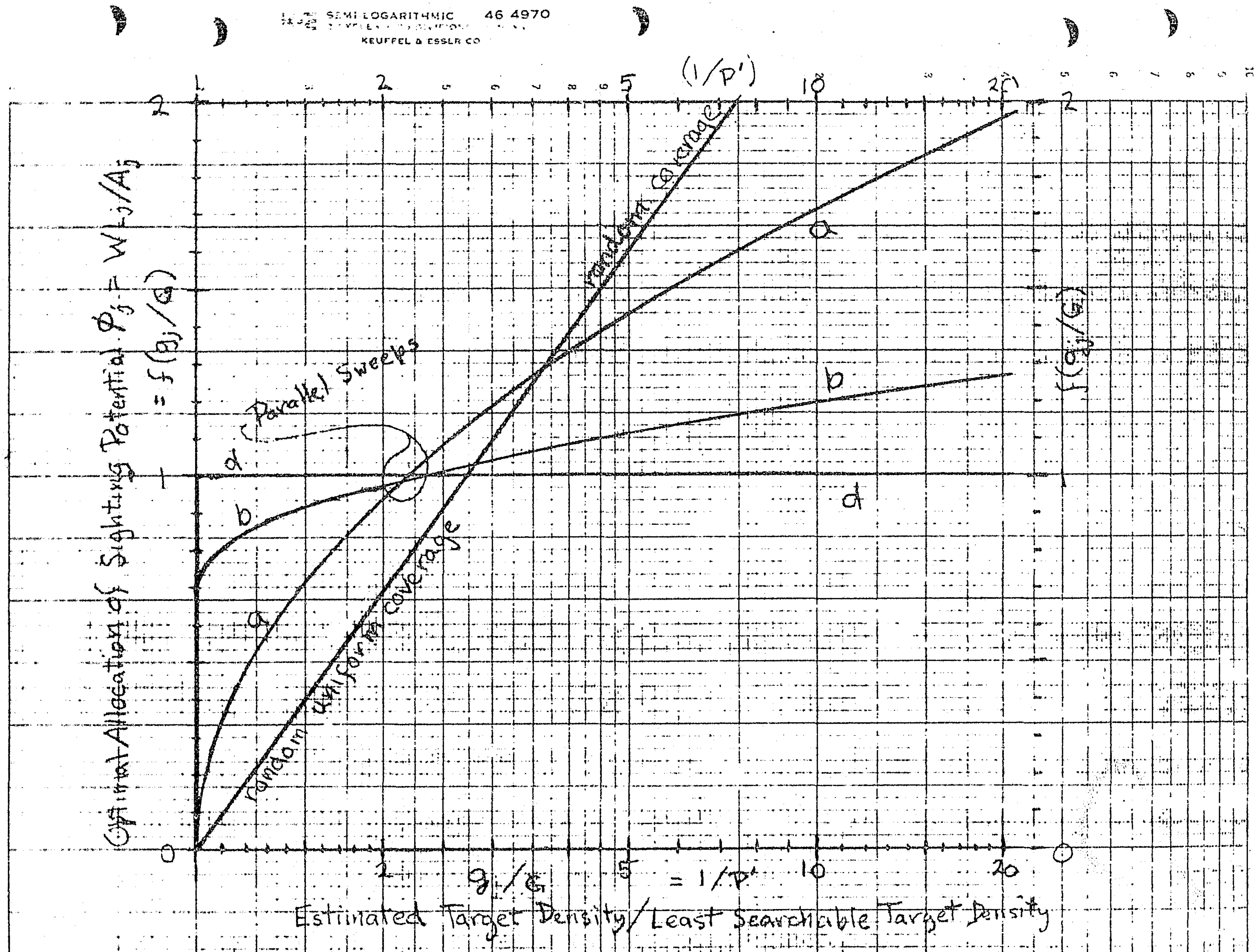


Fig.10. Inverse function $f(1/P') = \phi$ for the cases of Fig.8, to be used in the calculation ... h effort when target probability density g is not uniform over A.

value of P' is unity, which occurs for $\phi = 0$. As long as the value of target density g(r) in some region is larger than the value of g elsewhere, it is best to concentrate on the high-g region until the search there has reduced the Baysian, a posteriori target probability density to a value equal to the g for the next most likely region. And, if the search effort is limited, some low-g areas will have to be left out entirely. In fact, the process of optimal search may be restated in terms of a sequence of decisions as to where the next quantum of search effort can be most productively used (see Charnes and Cooper 1958 and Dobbie 1968, for example).

When false targets (call them ghosts) are present as well as the single target looked for, the analysis becomes much more complicated, and only a few cases have been worked out in detail. The results depend strongly on the number and nature of the ghosts and on the search strategy regarding them. The ghosts may be caused by sporadic malfunctioning of the detection equipment (among which may be included some of the radar ground and sea clutter and the reverberation in sonar equipment), in which the delay required to establish the contact as false may be quite short. Or the ghost may be a definite object (such as a sunken wreck or a "second-time-round" echo from an islet) that would take some time to verify as a ghost but, once verified, could be mapped so reverification would not be needed. Or the ghost could be mobile (as with a friendly ship or a whale) that would require reverification each time a contact was made.

Many strategies could be devised for dealing with these false targets. In regions where the ghosts are chiefly of the reverberation type, one might decide to limit the effort spent on any individual contact to a time longer than the average time needed to verify that the contact is really ~~the true~~ a false contact. An optimal search plan has been worked out by Stone, Stanshine and Persinger (1972) for a specialized case of this kind of search strategy. The results show the greatly increased complexity of calculations required to reduce the theory to practice. One is tempted to assume that the effect of such ghosts is to dilute the search effort required, without the ghosts, by a factor proportional to the estimated ghost density.

When the ghosts are actual targets, though false ones, the searcher may be forced to follow up each new contact for as long as it takes to determine whether it is true or false. Here also the resulting formulas (see Stone, et al., 1972 again) are quite difficult to apply. In the cases where the ghosts are stationary and mappable, a sequential procedure has been worked out by Dobbie (1973). An example of this procedure, for the simplest possible situation, will be given in the next subsection.

It should be noted that in many of the cases involving false targets the more feasible criterion for optimal search strategy appears to be the minimization of total effort (including that used to verify that a contact is a ghost) expected to be used to find the target, rather than the maximization of the detection probability for a given search effort. Indeed, in some cases, the two criteria may lead to different strategies (see Dobbie, 1973).

6.42 Applying the Formula.

Using procedures (23) in all their generality has disadvantages. First, it is seldom that one's a priori knowledge of the whereabouts of the target is good enough to enable one to specify target density g(r) in detail over all of A. Often we know only that it is within A; then the search coverage should be uniform over A. In some cases we can divide A into two subareas, with the target being rather more likely in one than in the other; in only a few cases is our a priori knowledge more detailed than this. It is thus useful to work out simple procedures to solve (23) for the two-subarea case.

Here the probability density g(r) is uniform within each of the subareas A_1 and A_2 (so the search density ϕ is uniform within each subarea) but g_1 differs from g_2 (so ϕ_1 will differ from ϕ_2). We can then reduce (23) to dimensionless terms by using as parameters and unknowns the following:

Ratios of areas; $\alpha_j = A_j/A$; $\alpha_1 + \alpha_2 = 1$

Probability that the target is in a subarea;

$$\gamma_j = A_j g_j = A\alpha_j g_j \; ; \quad \gamma_1 + \gamma_2 = 1$$

Minimum searchworthy probability of presence; $\lambda = AG$ (24)

Optimal specific search coverage of a subarea;

$$\Phi_j = \alpha_j \phi_j = WL_j/A = \alpha_j f(g_j/G) = \alpha_j f(\gamma_j/\alpha_j\lambda)$$

$$\Phi_1 + \Phi_2 = \Phi = WL/A$$

Optimal probability of finding target in a subarea;

$$\mathcal{P}_j = \gamma_j P[f(\gamma_j/\alpha_j\lambda)] \quad ; \quad \mathcal{P}(\Phi) = \mathcal{P}_1 + \mathcal{P}_2$$

The values of Φ_j and $(\mathcal{P}_j/\lambda)$, as functions of γ_j/λ are displayed in nomogram form in Figs. 11 to 14, for the cases of visual search in parallel sweeps and for random coverage, any detection means. The specific formulas for the two cases shown are obtained by referring to Eqs.(16) and (18);

For parallel sweeps, visual sighting

$$P(\phi) = \operatorname{erf}(\tfrac{1}{2}\sqrt{\pi}\,\phi) \; ; \quad P'(\phi) = \exp(\tfrac{1}{2}\tfrac{\pi}{4}\phi^2)$$

$$f(1/P') = (2/\sqrt{\pi})\sqrt{\ln(1/P')} = \phi$$

For uniform coverage of random sweeps per subarea (25)

$$P(\phi) = 1 - e^{-\phi} \; ; \quad P'(\phi) = e^{-\phi}$$

$$f(1/P') = \ln(1/P') = \phi$$

Parallel Sweeps, Visual Search. Determination of λ.

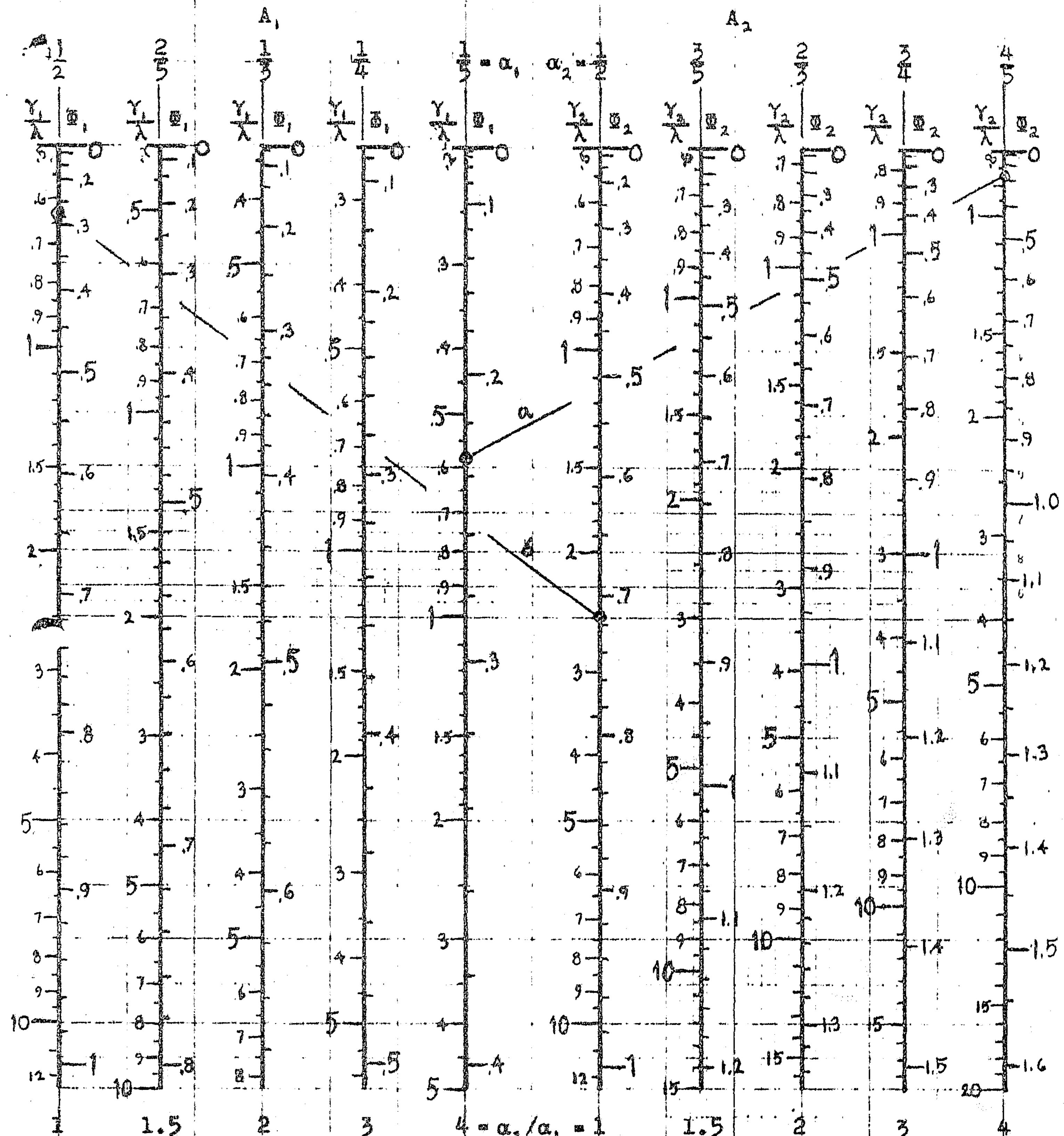


Fig.11. Nomogram for calculation of optimal allocation of Φ between two areas, for curve a of Figs.8 and 10.

Parallel Sweeps, Visual Search. Determination of ρ.

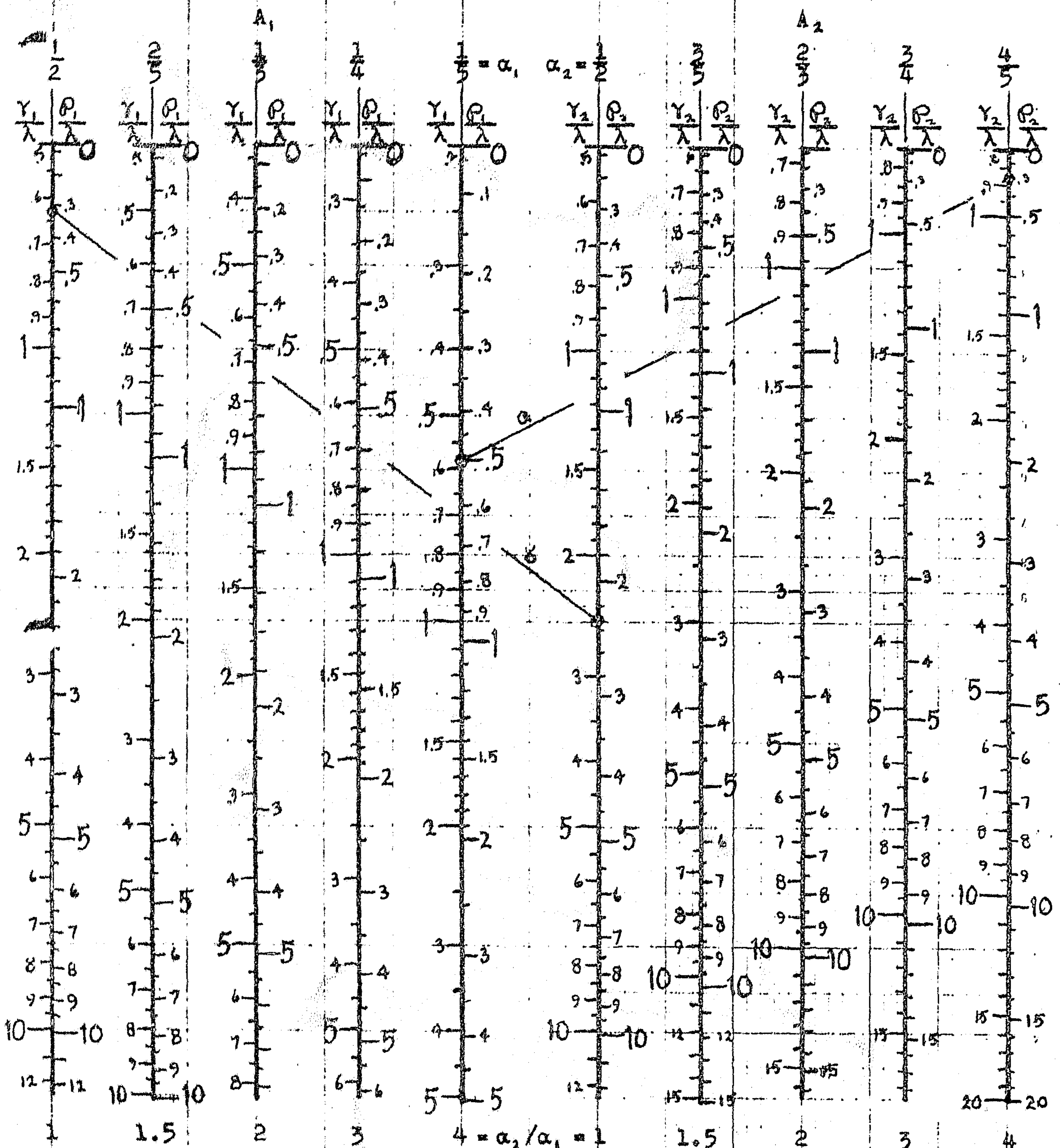


Fig. 12. Nomogram to be used with Fig. 11.

These are the functions used to compute the scales of Figs. 11 to 14.

The left-hand half of each chart corresponds to the smaller subarea, which we call A_1; the right-hand half goes with A_2. The first charts (Figs. 11 or 13) serve to determine λ, given the total search effort $WL = A\Phi$ that can be expended. Suppose we have guessed that the target has a chance γ_1 of being in subarea $A_1 = \alpha_1 A$ of the area A to be searched, and that it has a corresponding chance $\gamma_2 = 1 - \gamma_1$ of being in the subarea making up the rest of A, $A_2 = \alpha_2 A$. We first choose the pair of columns corresponding to the relative sizes of the subareas, given by the values of α_1 and $\alpha_2 = 1 - \alpha_1$.

Next we determine the ratio γ_2/γ_1 of the probabilities of presence in the two subareas. The γ_j/λ scales are logarithmic, so moving a line between the columns parallel to itself preserves the ratio of the γ's. For each value of γ_j/λ on the scale to the left of each column there is a corresponding value of Φ_j on the scale to the right. We slide the line parallel to itself until the two values, Φ_1 and Φ_2, picked out at the two ends of the line, add to equal $\Phi = WL/A$, the prescribed specific search effort.

Two examples are shown in Fig. 11. In case a we have guessed that the target has a probability $\gamma_1 = 0.4$ of being in the small subarea $A_1 = 0.2A$ and therefore that the chance of its being in the remaining $A_2 = 0.8A$ is $\gamma_2 = 0.6$. We also have decided that we can only spend a total effort $WL = 0.5A$ on the search. The ratio of the γ's is 1 to 1.5, so we set a ruler on $\gamma_1/\lambda = 1$ on the $\alpha_1 = 1/5$ column and on $\gamma_2/\lambda = 1.5$ on the $\alpha_2 = 4/5$ column and move it parallel to itself until the sum of the corresponding Φ's equals 0.5. This occurs at the two ends of the line a, for $\gamma_1/\lambda = 0.58$, $\Phi_1 = 0.235$ and $\gamma_2/\lambda = 0.87$, $\Phi_2 = 0.265$. Thus we must spend nearly half (0.47) of our search effort in the smaller area A_1. Note that if the available search effort were less than 0.26A the right-hand end of the parallel line would come above the top of the right-hand column, indicating that Φ_2 must be zero and that A_1 gets all the search effort, even though the chance of finding the target in A_2 is 1.5 times the chance of finding it in A_1. With such a small available effort, it is better to spend it all in the smaller area, where the probability density is greater.

Since we assumed γ_1 to be 0.4, λ must then be (0.4/0.58) = 0.69. To check we divide $\gamma_2 = 0.6$ by 0.87 and again get 0.69. To find the predicted probability of detection we turn to Fig.12 and draw the same line, between $\gamma_1/\lambda = 0.58$ and $\gamma_2/\lambda = 0.87$, between the same two columns. The probability scales on these columns show that $\mathcal{P}_1/\lambda = 0.5$ and $\mathcal{P}_2/\lambda = 0.3$. Having already found that $\lambda = 0.69$, we determine that the chance $\mathcal{P}_1$ of finding the target in A_1 is 0.34 and that of finding it in A_2 is $\mathcal{P}_2 = 0.21$, with a total chance of finding the target as 0.55.

Example b is for two equal subareas ($\alpha_1 = \alpha_2 = 0.5$) with the target guessed to be 4 times as likely to be in A_2 as in A_1 ($\gamma_1 = 0.2$ and $\gamma_2 = 0.8$). We have available this time a total search effort WL = A ($\Phi = 1$). Setting our ruler on $\gamma_1/\lambda = 1$ and $\gamma_2/\lambda = 4$ and moving it parallel we find that $\Phi_1 + \Phi_2 = 1$ at the ends of line b, for $\gamma_1/\lambda = 0.625$, $\Phi_1 = 0.28$ and $\gamma_2/\lambda = 2.5$, $\Phi_2 = 0.72$. Here we had better devote 3/4 of our search effort to the more likely area. Since $\gamma_1 = 0.2$ and $\gamma_1/\lambda = 0.625$, we have $\lambda = 0.32$, which can be checked, for 0.8/2.5 = 0.32. We note that if the total search effort is less than 0.66A then Φ_1 would be zero. In this case if WL is less than about (2/3)A, it is best to spend all of it in the more likely half of A. Turning to Fig. 12, the line points

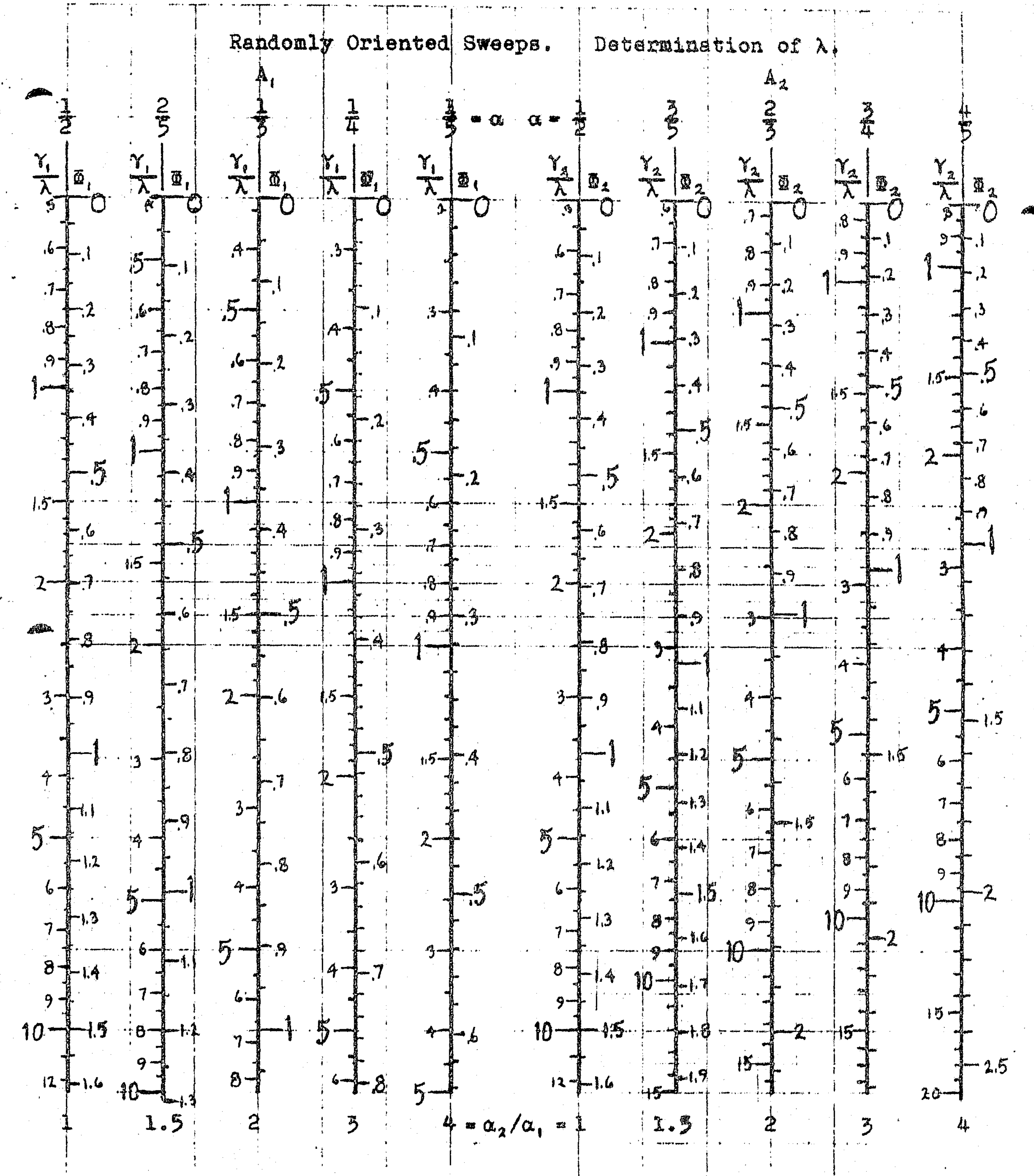


Fig.13. Nomogram for calculation of optimal allocation of Φ between two areas, for uniform, randomly oriented sweeps.

Randomly Oriented Sweeps. Determination of $\mathcal{P}$.

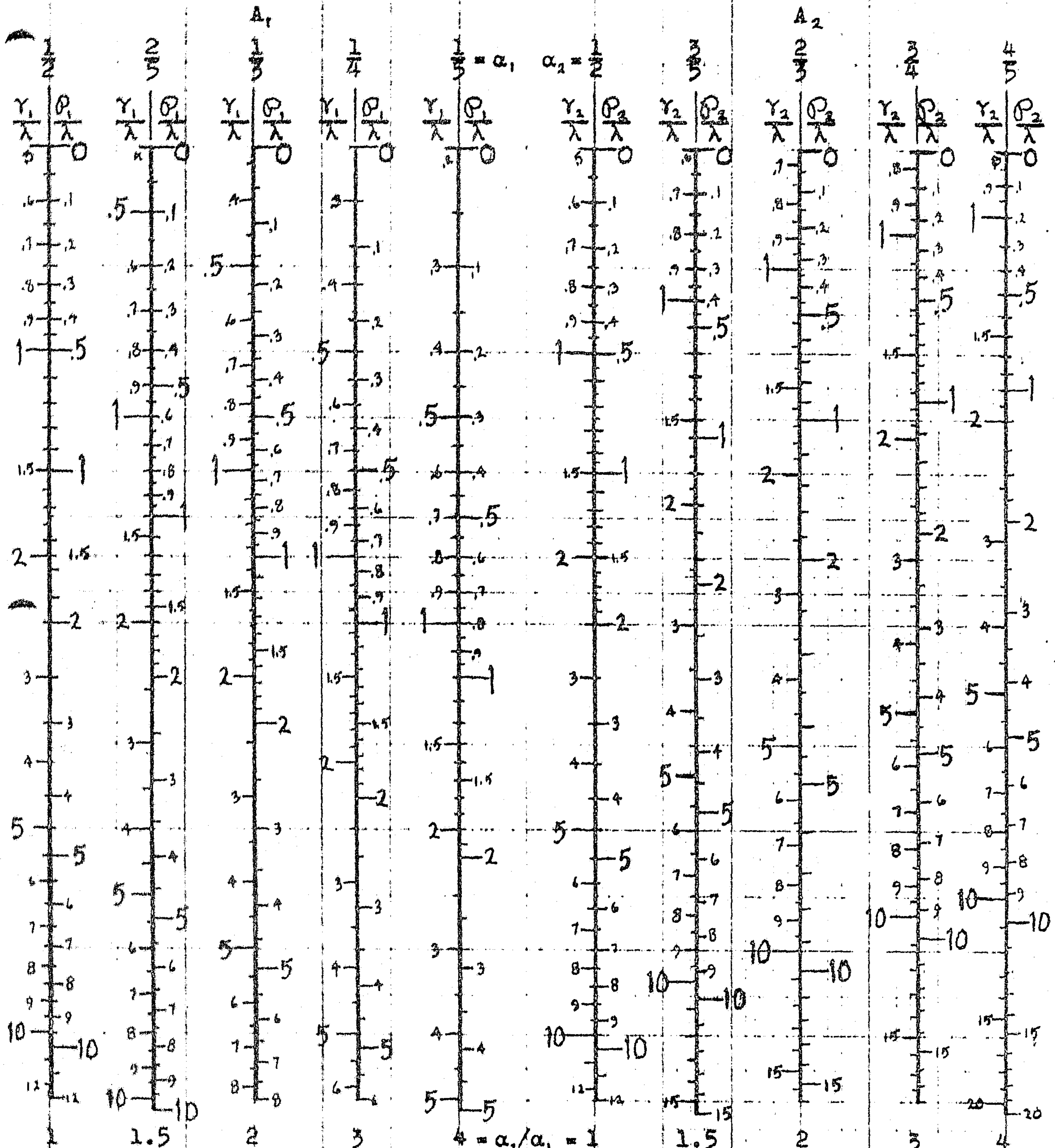


Fig.14. Nomogram to be used with Fig.13.

to $\mathcal{P}_1/\lambda = 0.31$ and $\mathcal{P}_2/\lambda = 2.32$. Since $\lambda = 0.32$, the probabilities are $\mathcal{P}_1 = 0.10$ and $\mathcal{P}_2 = 0.74$, with total probability $\mathcal{P} = 0.84$.

These examples are for parallel sweeps, visual search. Figs. 13 and 14 are for the more usual case of uniform coverage of each subarea by randomly oriented sweeps, and any form of detection equipment (the only effect the equipment has on the results is in its determination of the sweep width W). As indicated in Figs. 8 and 10, the probability of detection is not as great as with parallel sweeps with the same detection equipment and same search effort. But unless one is precise in traversing the parallel sweeps, the actual results are likely to be nearer the random sweep results than any of the parallel sweep curves.

One can fairly quickly work up curves for any particular case, from these nomograms. As examples Fig. 15 shows two pairs, to compare parallel, visual sweeps with random sweeps, for two different sets of α's and γ's. In the first case, the two to the left, we have a large difference in areas, $\alpha_1 = 0.2$ and $\alpha_2 = 0.8$, and a lesser difference in probabilities of presence, $\gamma_1 = 0.4$ and $\gamma_2 = 0.6$ (so the probability density of presence $g_1 = \gamma_1/\alpha_1 = 2$ in A_1 is larger than $g_2 = 0.75$). Thus the search starts in A_1, though not much effort is needed there before it begins to be worth while to begin searching in the larger A_2. The right-hand pair of curves are for somewhat more equal subareas, but a 2 to 1 ratio of probability of presence ($\alpha_1 = 0.4$, $\gamma_1 = 0.67$; $\alpha_2 = 0.6$, $\gamma_2 = 0.33$).

Note that the lower set of curves, for random sweeps display a quite similar pattern to the upper set, for parallel visual sweeps, but that the probability of success, for the same search effort is about 15 percent smaller for the random sweep cases.

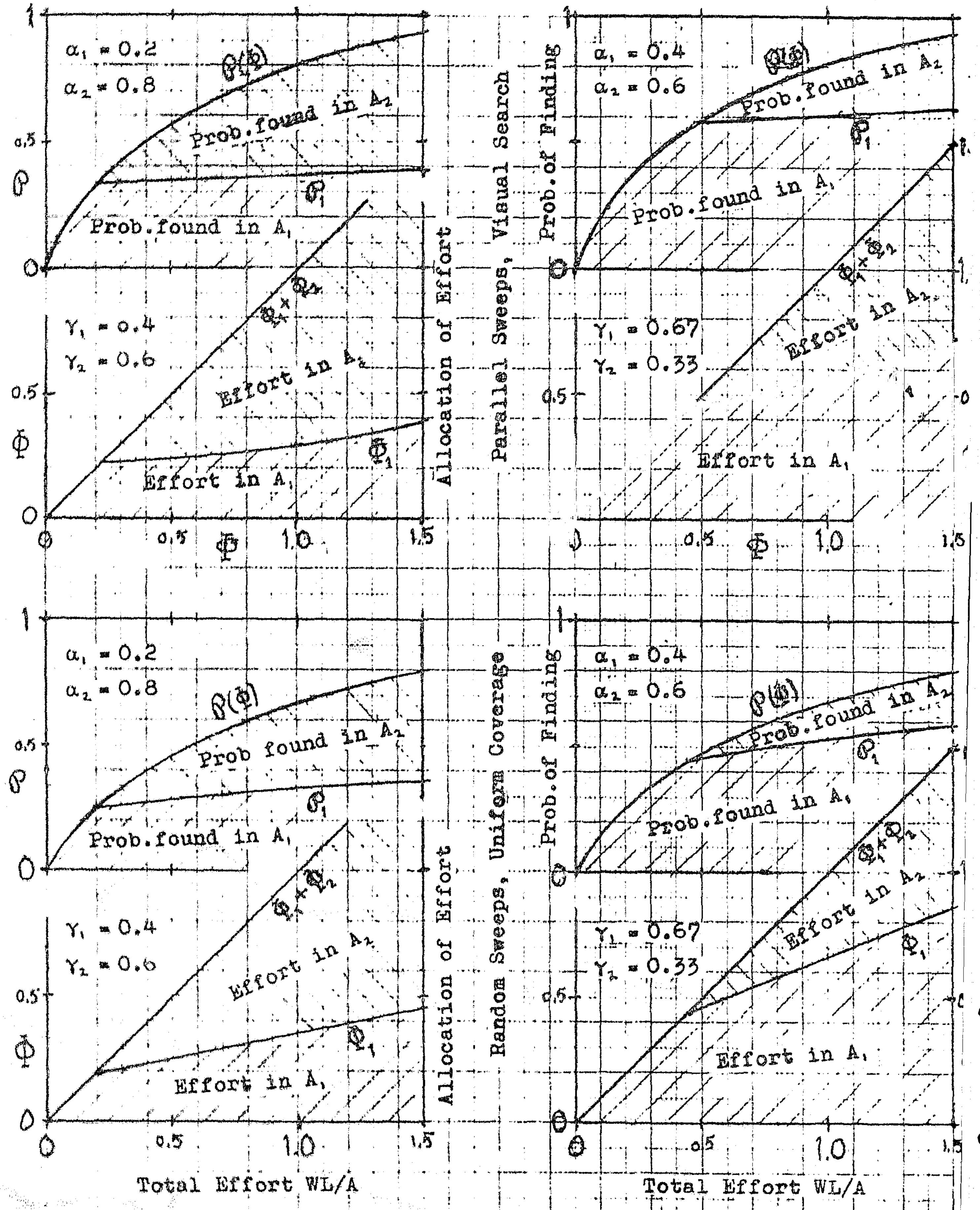


Fig.15. Examples of use of Figs.11 to 14.

The right-hand pair of curves, for the more nearly equal areas, shows that the less likely area is not touched until WL equals nearly A/2, but that by the time available effort has reached 3A/2 the search effort in the two areas is nearly equal(though the probability of detection in A_2 is still considerably smaller than that in A_1, further search in A_1 would not improve matters).

When the a priori estimates of the target's presence are detailed enough to require the separation of A into more than two subareas, a computational procedure to use with a minicomputer or a loglog slide rule can be developed for the case of random sweeps. Referring to Eqs.(23), (24) and (25), proceed as follows:

1. Having divided area A into N subareas, with area fractions $\alpha_j = A_j/A$ and target presence probabilities γ assigned to each, one rank-orders the areas in descending order of $\gamma_j/\alpha_j = g_jA$, starting with the subarea having the largest γ/α as A_1, and so on, so that $\gamma_{j-1}/\alpha_{j-1} \geq \gamma_j/\alpha_j$. Of course

$$\sum_{j=1}^{N}\alpha_j = 1 \quad \text{and} \quad \sum_{j=1}^{N}\gamma_j = 1$$

We then compute and tabulate the two sets of limits,

$$K_n = \sum_{j=1}^{n} \alpha_j \ln(\gamma_j/\alpha_j) \quad (n = 1,2,3,\cdots,N)$$

$$L_n = K_n - \Big[\sum_{j=1}^{n}\alpha_j\Big]\ln(\gamma_{n+1}/\alpha_{n+1})$$

Limits L_n form a monotonically increasing function of n.

2. When the total available specific search effort $\Phi = WL/A$ is less than $L_1 = \alpha_1\ln(\gamma_1\alpha_2/\gamma_2\alpha_1)$ search should be concentrated solely in subarea A_1. The probability of detection of the target (in the only searched A_1) is

$$\wp = \gamma_1(1 - e^{-\Phi/\alpha_1}) = \gamma_1(1 - e^{-WL/A_1})$$

3. When $L_{n-1} < \Phi = WL/A < L_n$ the search is to be in the subareas $A_1, A_2, \cdots, A_n$ only, with the search effort in A_j

$$A\Phi_j = A\alpha_j\left[\ln(\frac{\gamma_j}{\alpha_j}) + (\frac{\Phi - K_n}{\alpha_1+\alpha_2\cdots+\alpha_n})\right] \quad (j = 1,2,\cdots,n)$$

This results from the equation $\Phi = K_n - (\alpha_1+\cdots+\alpha_n)\ln\lambda$ so that $\lambda = \exp[(K_2 - \Phi)/(\alpha_1+\cdots+\alpha_n)]$. Therefore we have

$\sum_{j=1}^{n}\Phi_j = K_n + \Phi - K_n = \Phi$

The probability that the target will be discovered in A_j during the search is then

$$\mathcal{P}_j = \gamma_j - \alpha_j\lambda = \gamma_j - \alpha_j \exp\left[\frac{K_n - \Phi}{\alpha_1+\cdots+\alpha_n}\right] \text{ so that}$$

$$\mathcal{P} = \sum_{j=1}^{n}(\gamma_j - \alpha_j\lambda)$$

4. When Φ is greater than L_N, the largest of the L's of Eq.(26), then all subareas of A are to be searched, with individual efforts given by

$$A\Phi_j = WL_j = A\alpha_j\left[\ln(\gamma_j/\alpha_j) + \Phi - K_N\right]$$

which results from the equation $\lambda = \exp(K_N - \Phi)$

The probability that the target will be discovered in A_j is

$$\mathcal{P}_j = \gamma_j - \alpha_j\lambda = \gamma_j - \alpha_j\exp(K_N - \Phi) \quad \text{so that}$$

$$\mathcal{P} = 1 - \exp(K_N - \Phi) \qquad (26)$$

Note that $K_N \leq 0$ and that $K_N = 0$ only when all ratios $\gamma_j/\alpha_j = g_jA$ are equal and thus all equal to 1. Note also that if the search effort were to be distributed uniformly (random orientation) over the entire area A, the probability of detection would be $1 - \exp(-\Phi)$. Therefore, when $K_N < 0$ (i.e., when the probability densities of presence of the target, $g_j = \gamma_j/\alpha_j$ are not all equal) the probability of success $\mathcal{P}$ is increased if the search effort is allocated according to Eqs. (26). For another kind of application, see Morse (1970).

It can be shown (see Dobbie, 1963) that this allocation produces a $\mathcal{P}$ that is the largest achievable for a given Φ; likewise that the Φ, distributed according to the formulas, is the smallest effort that can achieve the resulting $\mathcal{P}$.

One example of the results is shown in Fig.16, for three subareas. The least likely area A_3 is neglected until the search coverage WL becomes greater than (3/4)A. The dashed line shows the probability of success if the search effort had been spread evenly over A. One would have to increase Φ by 20% to get an equal $\mathcal{P}$.

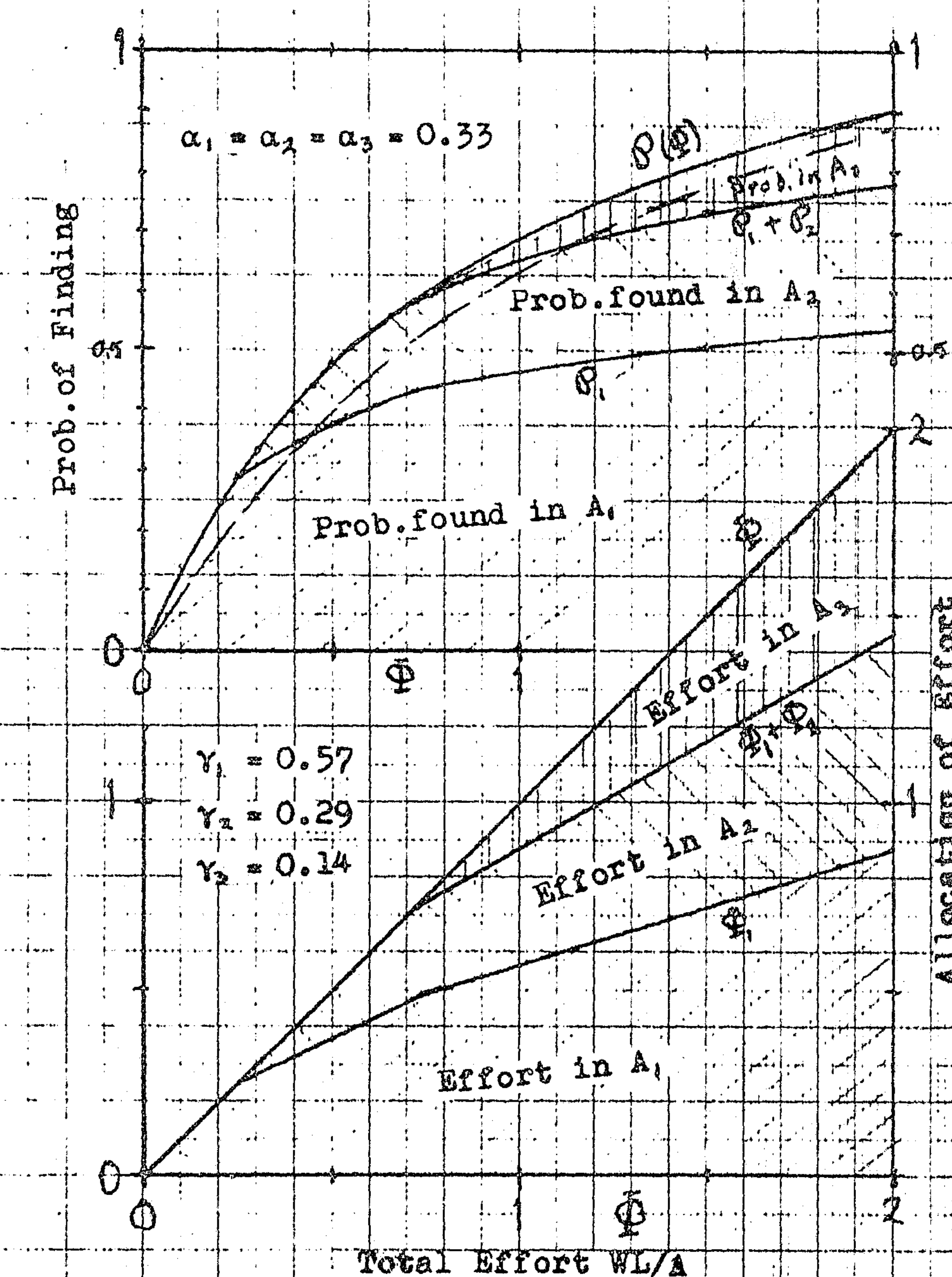


Fig.16. Allocation of search between three subareas.

An example of the effect of false targets has been worked out by Dobbie (1973), for the very simple case of two areas, thus analogous to the examples given in Figs.(11) to (15). Expressed in the nomenclature of Eq.(24), the area A is divided into two equal parts, so $\alpha_1 = \alpha_2 = 0.5$. There is to be only one false target, which is stationary, so if it is once located its presence can thereafter be ignored. We are supposed to know whether it is in A_1 or A_2, but we do not know its whereabouts in the subarea. As before, we assume we know the probability γ_1 that the true target is somewhere in A_1, and thus the probability $\gamma_2 = 1 - \gamma_1$ that it is somewhere in A_2. Also as before we assume A_1 to be the subarea with the larger probability of presence of the target, $\gamma_1 \geq \gamma_2$. Search effort is given in terms of the specific effort $\Phi_s = WL_s/A$.

In this case Dobbie minimizes the expected specific effort $\Phi_s + \Phi_e$ required to find the true target, where Φ_s is the specific effort spent in search and Φ_e is that spent in determining whether the contact is true or false (and we assume that the expected time required for each determination, whether it be the true or false target investigated, is unity). Random-uniform search is assumed, so the probability of making a contact, on either the false or the true target, by application of search density $\phi_j = WL_j/A_j$ in A_j is $1 - e^{-\phi_j}$, as in Eq.(25). According to the total specific effort $\Phi_m = WL_m/A$ available, it is allocated sequentially in searching over A_1 or A_2, or in verifying a contact, as long as there is available effort, in the order given by the following scenario, which is presented in a form convertable to a flow chart.

There are three possibilities:

Ia) If the false target is somewhere in A_1 and $1 > \gamma_1 > 0.6418$, search A_1 until either A or B occurs, which ever comes sooner;
- A) A contact is made. Then spend the requisite specific effort $\Phi_c = 1$ to determine whether it is the true or false target.
  - a) If it is the true target, go to 1 below.
  - b) If it is the false target, record its position and resume the search of A_1 until either α or β occurs, whichever comes sooner:
    - α) Another contact is made. Go to 1 below.
    - β) A total of $\Phi_s = \frac{1}{2}\ln(\gamma_1/\gamma_2)$ of search effort has been expended in A_1. Then go to 2 below.
- B) An amount $\Phi_s = \frac{1}{2}\left[\ln(\gamma_1/\gamma_2) - 0.5831\right]$ of search effort has been expended in A_1 without obtaining a contact. Then go to 2 below.

Ib) If the false target is somewhere in A_1 and $0.6418 \geq \gamma_1 > 0.5$, search A_2 until either A or B occurs, whichever comes sooner;
- A) A contact is made. Go to 1 below.
- B) An amount $\Phi_s = \frac{1}{2}\left[0.5831 - \ln(\gamma_1/\gamma_2)\right]$ of search effort has been expended in A_2 without contact. Then go to 2 below.

II) If the false target is somewhere in A_2 and $1 > \gamma_1 > 0.5$, search A_1 until either A or B occurs, whichever comes sooner;
- A) A contact is made. Go to 1 below.
- B) An amount $\Phi_s = \frac{1}{2}\ln(\gamma_1/\gamma_2)$ of search effort has been expended in A_1 without contact. Then go to 3 below.

1) This is the true target. Stop the search.

2) Search the whole area $A = A_1 + A_2$ uniformly until a contact is made. If the contact is in A_2, go to 1. If the contact is in A_1 expend the requisite effort $\Phi_e = 1$ to determine whether it is the true or the false target.
- 2.1) If it is the true target, go to 1.
- 2.2) If it is the false target, record its position and resume the search in A_1 only, until 2.2.1 or 2.2.2 occurs, which ever comes sooner;
  - 2.2.1) A contact is made. Go to 1.
  - 2.2.2) An additional amount $\Phi_s = 0.2915$ has been expended in A_1 without contact. Then go to 4 below.

3) Procede as in 2, but interchange A_1 and A_2 in the instructions.

4) Resume the search uniformly over the whole area $A = A_1 + A_2$ until a contact has been made. Then go to 1.

The probability of detection of the true target, as a function of the available specific effort E, expended in accordance with this scenario, is not given explicitly by Dobbie, but the procedure will ensure that, on the average, the effort expended will be the least amount required to attain that probability. Since this, almost the simplest of search allocation problems involving false targets, gives rise to operating rules that would be difficult to follow in the heat of an actual search, it may be questioned whether precise analysis of more complex situations would be more an admirable mathematical exercise than a practical aid in actual searches. One can hope that approximate solutions can be developed that will be simpler to carry out in practice.

6.5 Target Motion.

The previous sections have assumed that the target moves slowly enough that it would have moved a negligible distance during the whole search operation. If there is no *a priori* knowledge of the whereabouts, in area A, of the target, nor of its direction of motion, this motion does not alter the fact that it may be anywhere in A (assuming that it cannot leave A). Thus the best procedure still is to provide uniform coverage of A. If the target can leave A, the area to be searched will have to be increased in size as the search proceeds. If this increase is a small fraction of A, it makes little difference in the organization of the search or in its outcome. On the other hand if the increase is equal to or greater than A by the time A has been searched over and if the target has not been found by then, it is unlikely that further search will be able to keep up with the expanding area of presence.

6.51 Target Position and Motion Unknown.

To justify these statements, it is convenient to use the differential equation governing the probability P of finding the target. The searcher, as in previous sections, is assumed to move with velocity v and to have a search width W. Referring to Fig.9b, the increase in the probability P(L) of having found the target, after a search path of length L, is equal to the increase in the area of coverage $da = W\,v\,dt$, divided by the area q, within which the target is likely to be, and multiplied by the probability $1 - P$ that the target is not yet found;

$$dP = (1 - P)(W/q)dL = (1 - P)(da/q) \qquad (27)$$

where $a = WL$ is the area already searched over by the time the path has reached length L.

If the target is confined within the area A and if initially it can be anywhere inside A, then target motion will not change its probability density of presence, which will be (1/q) at any instant. If, in addition, the search is random-uniform, then the area q in Eq.(27) will be practically equal to A (as long as W^2 is small compared to A); possible target motion within a confined area A cannot change its probability density of presence. The equation and its solution are then

$$\frac{dP}{P-1} = -\frac{W}{A}dL \;; \quad P(a) = 1 - e^{-WL/A} = 1 - e^{-a/A} \qquad (28)$$

identical with Eq.(18).

On the other hand, if the target can cross the perimeter of the area A, the area of possible presence, q in Eq.(27), increases with time. In this subsection we suppose the target is not aware of the searcher and that its motion is randomly oriented. If it happened to be on the perimeter of A, in only half the cases would its motion take it outside A, and the average distance it would penetrate beyond A in time dt would be

$$(u\,dt/2\pi)\int_0^\pi \sin\theta\, d\theta = (u/\pi)dt = (u/\pi v)dL = (u/\pi vW)da$$

where u is the estimated mean target speed of target motion, v is the speed of the searcher and W is his search width. This "leakage" produces a gradual enlargement of the area of presence of the target, over the initial value A, as though the various possible positions of the target were molecules in a gas, with

mean speed u. The gas expands with a mean velocity (u/π) normal to the boundary. In fact the expected enlargement of the area is $dq = \frac{u}{\pi} s\, dt = \frac{us}{\pi vW} da$

where s is the length of the perimeter of q. If the initial area A is circular or square, s is equal to the perimeter S of A times $\sqrt{q/A}$. Even if the length of A is twice its width, the formula $s \simeq S\sqrt{q/A}$ is approximately correct as long as q is less than twice A.

Therefore the area of presence of the target q, after area a has been searched over in a random-uniform manner, is approximately equal to the solution of the differential equation

$$\frac{dq}{\sqrt{q}} = 2\gamma \frac{da}{\sqrt{A}} \quad \text{or} \quad q = A\left(1 + \gamma\frac{a}{A}\right)^2 \quad \text{where} \quad \gamma = \frac{uS}{2\pi vW}$$

Inserting this into Eq.(27) we obtain the probability P(a) of detection of the target after random-uniform search effort $WL = a$ of an area initially of magnitude A, when the target is initially anywhere within A and has an estimated, randomly directed speed u (and is not confined within A)

$$P(a) \simeq 1 - \exp\left[\frac{-a/A}{1 + \gamma(a/A)}\right] \;; \quad a = WL \qquad (29)$$

This differs from Eq. (28) by the term in the denominator of the exponential, resulting from the "leakage" of the moving target into the region outside A. It is a valid approximation as long as factor $\gamma = (u/2\pi v)(S/W)$ is small, which assumes that ratio (u/v) of estimated average target speed to searcher speed is no larger than the ratio (W/S) of search width to perimeter of A. In fact if $\gamma \simeq 1$, P(a) ceases to increase soon after a becomes equal to A, after which area q expands faster than the search can catch up. However if (W/S) for a single searcher is smaller than u/v, γ can be reduced in value by employing more than one searcher. For m independent searchers, each following a uniform-random path, W in the formula is changed to mW and thus γ is changed to (γ/m).

To see what degradation is produced by this possible "leakage" of the target outside initial area A, we tabulate P(A), the probability of detection after a total area $mWL = A$ has been searched over, for different values of γ.

Table 1.

γ	0	0.1	0.2	0.3	0.4	0.5
P(A)	0.632	0.597	0.565	0.537	0.510	0.487
a_1/A	1.000	1.111	1.250	1.429	1.667	2.000

The third line measures the area $a_1 = mWL_1$ that must be searched over in order that the probability of detection $P(a_1)$ equal the value 0.632 for $\gamma = 0$ and $a = A$. Further increase of a beyond a_1 of course produces further increases of P(a), but these further gains are made at the cost of disproportionately large efforts. The gain in P(A) by dividing the search efforts among m searchers comes in the fact that γ becomes γ/m, because each searcher needs to search only area A/m (provided the m paths ,between them, cover A uniformly) and the search is completed in (1/m)'th the time, so the target has less time to "leak out".

An exact analysis of target motion on a regularly patterned search (such as the parallel sweeps of Fig.6) has not yet been worked out. For comparison, as an opposite limit from the random-uniform case just presented, we can look at the idealized case of using definite-range equipment in parallel sweeps spaced $S = W$ apart, so as to leave no unsearched area between

sweeps. If the target is at rest somewhere within A, the probability of detection by the time the path length has reached $L = a/W$ is given by Eq.(6) and by curve d of Fig.8 $(P = a/A)$.

First, we assume that the target is in motion, with a randomly directed average speed u, but that it is confined to motion within the initial area A. In this case the only way the target can "leak out" is into the area a, that was assumed to have been completely swept. Examination of Fig.17 indicates that when the area is completely swept, so that $a = LW = A$, the regions where the target could have leaked back (the cross-hatched areas) have an area

$$q(A) = \frac{nu}{\pi v} D^2 = \frac{uD}{\pi v W} A \quad \text{since} \quad n = \frac{C}{W} \quad \text{and} \quad A = CD$$

within which the target may still reside, unfound (we assume it takes n parallel sweeps to completely cover A).

This leaked-back area is approximately proportional to the swept area so that, at the stage when area a has been swept (as shown in Fig.17) the area within which the target may still be (if it has not yet been discovered) is

$$q(a) = A - (1-\mu)a \quad \text{where} \quad \mu = \frac{u}{\pi v}\frac{D}{W}$$

Finally, inserting this into Eq.(27) results in

$$\frac{dP}{1-P} = \frac{da}{A-(1-\mu)a} \quad \text{or} \quad P(a) = 1 - \left[1-(1-\mu)\frac{a}{A}\right]^{1/(1-\mu)} \qquad (30)$$

for the case where the target is constrained to move inside A.

The probability of detection when $a = A$ (when the search would have been complete if there were no target motion) is not unity but $P(A) = 1 - \mu^{1/(1-\mu)}$

This is tabulated for a few values of μ;

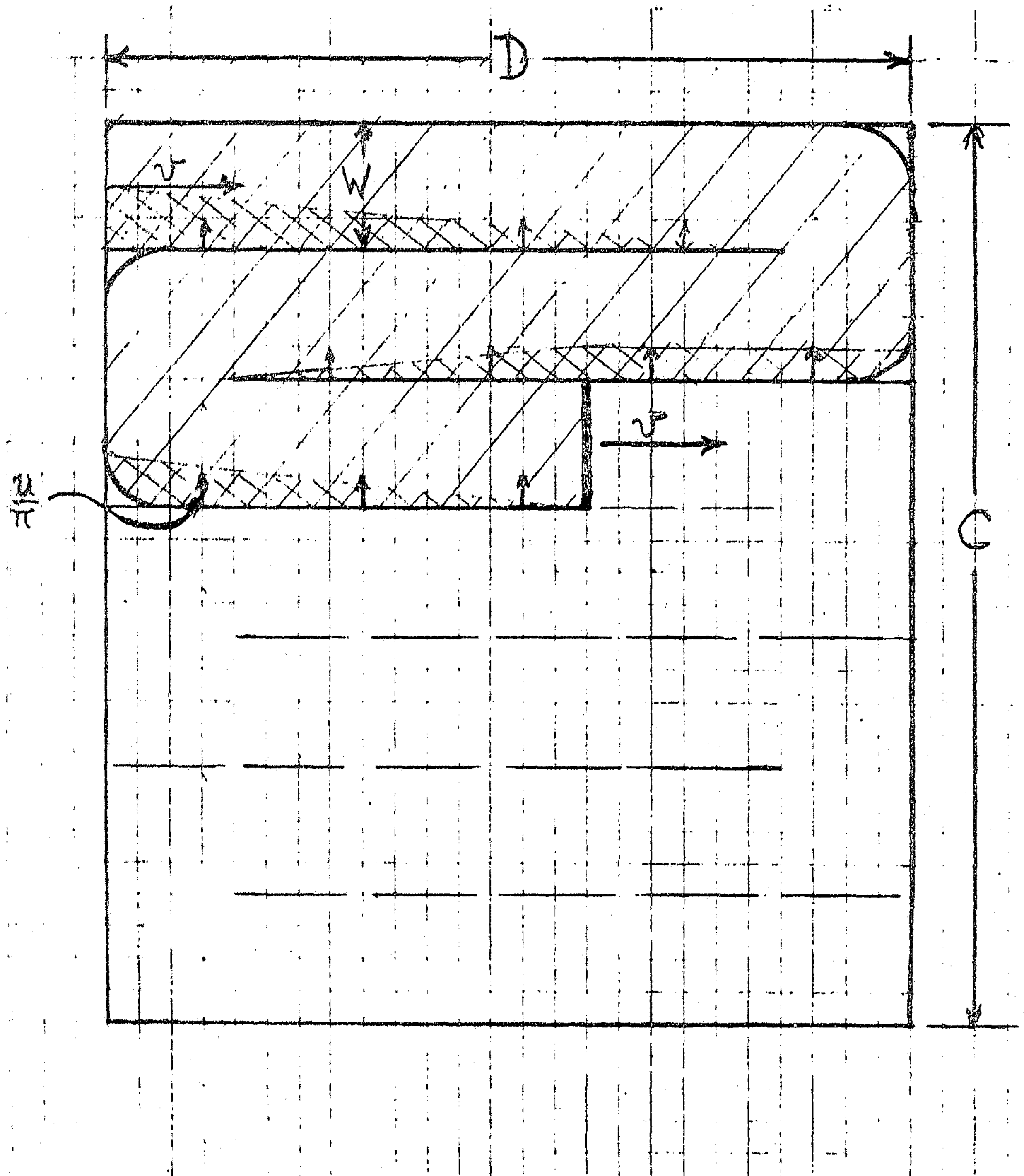


Fig. 17. Tight search coverage of area $A = CD$. Shaded area is a, the portion completely searched if target is at rest. If target has average speed u in a random direction it can leak back into cross-hatched area b. Area of possible presence of target is thus $A - a + b$, if target cannot cross perimeter of A.

Table 2.

μ	0	0.05	0.10	0.15	0.20	0.25
P(A)	1.000	0.957	0.923	0.893	0.866	0.843

As with the random case, the analysis is valid when constant $\mu = (u/\pi v)(D/W)$ is small. Better results can be achieved by using m searchers, in which case each searcher needs to cover only area A/m and constant μ becomes $(u/\pi v)(D/mW)$. In an actual search the detection will not have the sharp cut-off of the definite range curve d of Fig.3, nor will the sweeps be the perfect pattern of Fig.17. Therefore the actual probability of detection for $WL = A$ will be somewhere between the P(A) of Table 2 and the $1 - e^{-1} = 0.632$ of Eq.(28) for random coverage. The difference between these limits is less when a is less than or is greater than A.

If the target is not prevented from crossing the perimeter of A, the bottom side of A (as shown in Fig.17) will be penetrated and the increase in searchable area, because of this side, when $a = A$, is D times the effective velocity u/π of leakage, times the duration $T = nD/v = A/vW$ of the search. This also increases linearly as the search progresses, so this addition to the area of q also is $(uD/\pi vW)a = \mu a$. The leakage out of each of the sides C of A is a stepwise approximation to a triangle with vertex at the upper corner and base, down a distance $C(a/A)$ from the top, of width $(u/\pi vW)a$, plus a rectangle of width $(u/\pi vW)a$ beyond the rest of side C. The added area on both sides, when (a/A) of the search has been completed, is thus $\nu[2 - (a/A)]a$, where $\nu = (uC/\pi vW)$. Therefore when the searched area is $a = WL$, the area within which the target may yet be is

$$q \approx A - a(1 - 2\gamma) - (\nu/A)a^2$$

where $\gamma = \mu + \nu = (u/\pi v W)(C + D) = (u/2\pi v)(S/W)$, S being the perimeter of A, as in Eq.(29).

The probability of discovering the target after area a has been searched over by the "ideal" coverage of Fig.17, is the solution of Eq.(27) with this new value of q inserted. It is

$$P(a) = 1 - \lambda\left\{\frac{1-\lambda[1-2\gamma+2\nu(a/A)]}{1+\lambda[1-2\gamma+2\nu(a/A)]}\;\frac{1+\lambda(1-2\gamma)}{1-\lambda(1-2\gamma)}\right\}^{\lambda} \qquad (31)$$

where $\lambda = 1/\sqrt{1-4\mu+4\gamma^2}$. To measure the effect of this leakage we tabulate P when A is a square (when $\gamma = 2\mu = 2\nu$) and when $a = A$, for different values of γ;

Table 3.

γ	0	0.1	0.2	0.3	0.4	0.5
P(A)	1.000	0.878	0.785	0.718	0.680	0.667

Comparison with Table 2, for the case when the target is kept inside A (for $\gamma = 2\mu$) shows that leakage over the perimeter of A produces a considerable reduction in the probability of detection. Of course if m searchers are used, moving accurately in line abreast, W becomes mW and γ becomes γ/m. If one has enough manpower, the search can be completed quickly enough so the effect of target motion can be minimized.

Of course Table 3 is for the perfect coverage of A implied in Fig.17. If the lateral range curve differs from d of Fig.3 and/or the sweeps are not exact, the detection probability P(A) will approach the lower limit given in Table 1, for random coverage.

Other patterns of parallel sweeps, with definite range equipment, will result in slightly different values of the upper limit of P(A), but the difference will not be large.

For example, the approximate analysis for a path that covers the perimeter first and then spirals in to the center yields results similar to those of Table 3, but with $\gamma \simeq (3uS/8\pi vW)$, roughly three quarters of the γ for Eq.(31). Covering the perimeter of A first is some improvement, but the leakage into the swept path still occurs. In any case these results are for the ideal case of definite range, perfectly aligned, parallel sweeps. It is safer to predict probabilities nearer those of Table 1, for random-uniform sweeps.

6.52 Crossover Barrier.

When the target motion is not randomly oriented, the search problem differs again. Only two specific cases have been analyzed sufficiently to yield results of practical utility. The simplest case of this sort is when the direction and magnitude of the target's motion is known, but its position is not.

Suppose an aerial search is to discover a ship that must pass through an ocean strait, of navigable width D, as shown in Fig.18a. If the ship is known to have a speed u as it passes through the strait, we can analyze the search path most easily by transforming to a coordinate system moving with the ship, as shown in Fig.18b. In these coordinates the most efficient search path will be series of parallel sweeps that transform back to coordinates at rest with respect to the ocean, as the angular figure 8 shown in Fig.18a. Note that the short, end legs are in a direction opposed to that of the target.

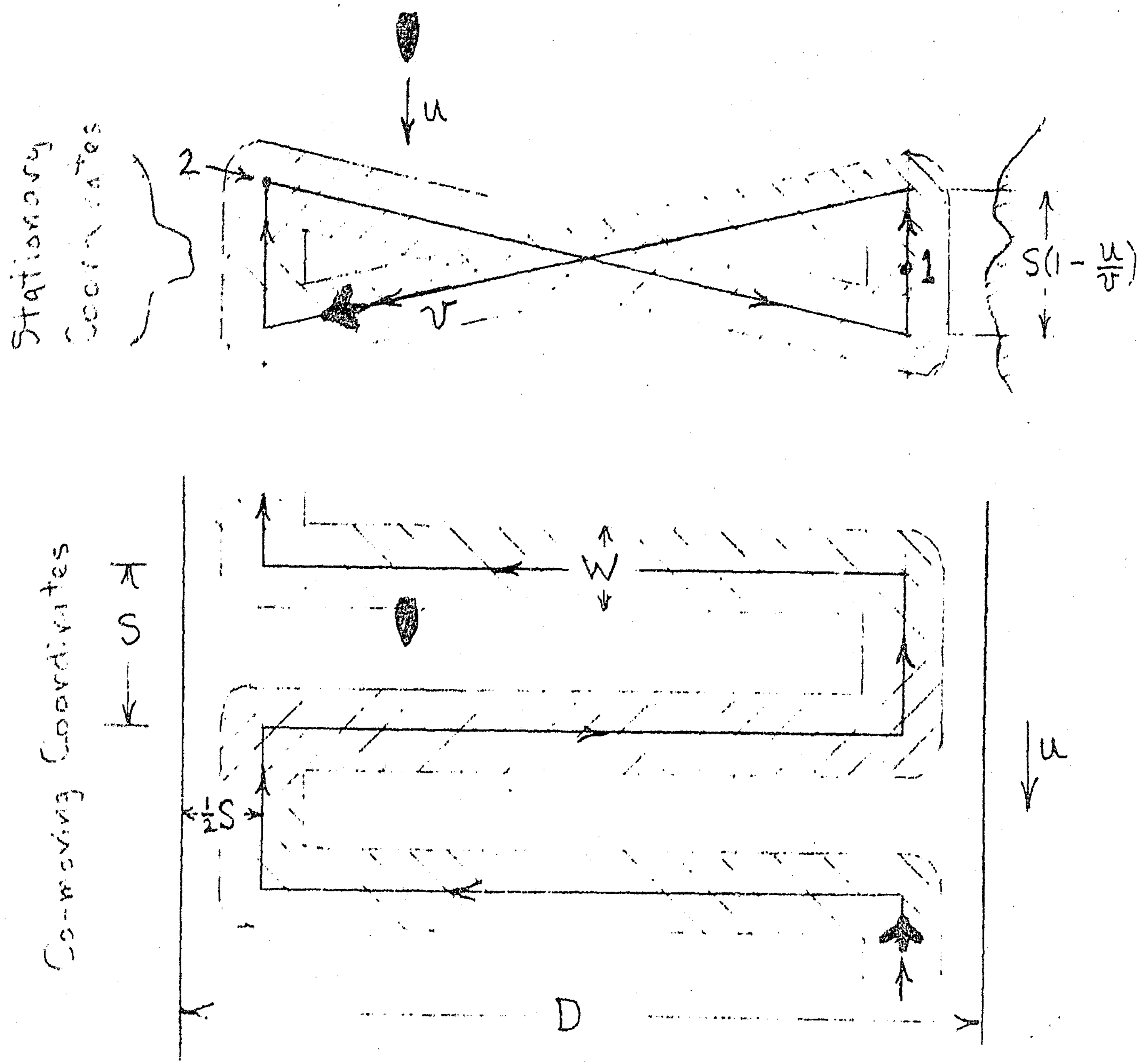


Fig.18 A cross-over barrier patrol to catch a ship passing through an ocean strait.

If this path is to close on itself, if the barrier patrol is to keep up with the motion of the target, the spacing S between parallel sweeps in co-moving coordinates must be related to the width of the strait, to the speed u of the target and to the speed v of the search plane. The time $T = 2D/v$ that it takes the plane to go across and back (we assume u/v is small enough so the length of the diagonal leg in Fig.18a is nearly equal to D; if not, correction can be made) must equal the length of time $T = 2S/u$ for the co-moving coordinates to move downward by two sweep spacings. Thus spacing S and the resulting sighting potential Z are given by the formulas

$$S = D(u/v) \quad ; \quad Z = W/S = (Wv/Du) \qquad (32)$$

This path, translated back to stationary coordinates, as shown in Fig.18a, is called a <u>crossover barrier patrol</u>.

If it is not known where, along the line D across the strait, the target is to pass, nor is it known, within a time $T = 2D/v$, when it is to pass, then the whereabouts of the target may be anywhere within an area 2SD in co-moving coordinates. Thus, <u>if</u> the barrier patrol is in operation when the target passes through the strait, the probability of detection is the probability $P(Z)$ that has been given in Eqs.(16) or (18) or shown in Fig.8, for parallel sweeps. There is a more complete discussion of this problem in Chapter 7 of Koopman, 1946.

If Z, as given in Eq.(32), is less than 1/2, i.e., if the ratio W/D is less than half the ratio u/v, between target speed and search plane speed, the probability P of detection will not be satisfactorily large. Several search planes should then be used if possible, either flying parallel courses a distance S/n apart (if n planes are used) or spaced in sequence along the same course, spaced in time. For example, if two planes are to be flown in sequence, the second plane should be started at point 2 in Fig.18a when the first plane is at point 1, halfway up the opposite vertical leg; in this manner the second plane's sweeps would come half way between those of the first plane, in co-moving coordinates. If the multiple sweeps are carefully flown, so that the n parallel sweeps are equally spaced in comoving coordinates, the effective sighting potential would be nWv/Du, and this value could be used as Σ to determine the probability $P(\Sigma)$ of detection.

Barrier patrols are useful in many other military, police and life saving operations.

6.52 Retiring Search Sweeps.

Another situation, not infrequently encountered, arises when the target is located exactly, at some instant, but the search is not able to start until a time T_o later. One has to assume that the target has moved during that time and, if the target's maximum velocity u is known and if there is no indication of the direction of its motion, at T_o it could be anywhere within a circle of radius uT_o. As the search progresses, this circle of presence continues to expand; so the search path, if possible, should be an expanding spiral, trying to cover this increasing area.

If the search effort is to be limited to the value $E(T) = WvT$, it must be within that part of the circle, of radius $R(T) = u(T_o + T)$, that the allocation rules of Eqs.(23) say should be searched. In most cases, when it is not known whether the target's actual speed is its maximum speed u, or zero, or something in between, the point of maximum probability of presence of the target would be at the origin, where the target had originally

been spotted. Therefore the search would begin at the center and spiral outward, with spacing between the arms prescribed by the density $\phi = W/S$ determined by Eqs.(23).

The analysis of this operation is still more "intrinsic" than that for a stationary target. We shall go through it using the formulas for random sweeps, partly because it is the only case for which the answers can be analytic and partly because it is unlikely that careful interpath spacing can be maintained in a spiral search, so it is safer to assume the less optimistic formulas. We consider the case at time T, when the allocated search effort WvT has been used up and the radius of the circle of presence of the target is $u(T_o + T)$. Looking back on the search, that started at the center at $t = 0$ and spiralled out, as the spiral passed through the radius $r < R(T)$ and effort $E(t) = Wvt$ has already been used up, the *a priori* estimate of the probability density of presence of the target there would then have been $g(r)$, which can be estimated for each value of r and T out to the value at which the search ends.

It is more convenient, and leads to easier generalization if the area of presence is not a circle, to change variables from radius to area. The area of presence $A(t)$ at time t after the start of the search and the area $q(t)$ inside the circle of radius r (the area already searched over by time t) are given by the formulas (note that this q is not the same as the q of Eqs. 27 to 31),

$$A(t) = \pi u^2(T_o + t)^2 \quad ; \quad q(t) = \pi r^2$$

Time during search can be measured in terms of search effort

$$E(t) = Wvt$$

so that these variables can be expressed in terms of dimensionless quantities

$$\left.\begin{array}{l} A(t) = \alpha(1+z)^2; \quad z = (t/T_o) = E(t)/\beta \; ; \; q(t) = \alpha x \\ \alpha = A(0) = \pi u^2 T_o^2 \; ; \quad \beta = WvT_o \end{array}\right\} \quad (33)$$

¶ Thus α is the area of presence of the target at the start of the search and β is the area that could have been searched with density $\phi = 1$ during the time T_o (which we can call the <u>delay time</u>). The interrelation between these quantities must be given in terms of the rules of search given in (23), only now they must include the fact that the probability density of presence $g(r)$ refers to the time t at which the plane was searching at the distance r from the origin.

Before we start the analysis, however, some salient points should be noted. When the search ends at time T the search effort WvT has been expended. But by this time the area of presence of the target has become $\pi u^2(T_o + T)^2$. Therefore, by the end of the search the mean sighting potential $\Phi = E/A$ has become $[WvT/\pi u^2(T_o+T)^2]$. This quantity increases with T for a while, but it reaches a maximum at $T = T_o$ and theafter declines. Its maximum value at $T = T_o$ is $\Phi_m = (W/4\pi u T_o)(v/u)$, inversely

proportional to T_o, a product of a ratio of areas and a ratio of velocities of target and searcher. This has two consequences: first, the sooner one can start the search the more efficient is the search coverage and, second, the first part of the search, during a time equal to the delay time T_o, is by far the most effective part of the search. Further search, beyond $T = T_o$, is chasing a widening circle of presence that has already gone too far to catch up with.

Returning to procedure (23), we first have to decide on a reasonable form for the probability of presence $g(r)$, at the instant of time t when the search has reached r and the area of presence has reached $A(t)$. Since we do not know the actual speed or course of the target, beyond its maximum speed u, we might assume an average distribution and let

$$g(r) = (2/A)\left[1 - (\pi r^2/A)\right] = (2/A)\left[1 - (q/A)\right]$$

with its maximum at $r = 0$, tapering off to zero at $q = A$. Since A is a function of t and therefore of z, g is a function of z. But since $q = \alpha x$ also is a function of search effort $z\beta$, we can say that A, g, and z are all functions of x. Their interrelations are given by (23).¶ We require that the density g, times $P'(\phi)$, the derivative of the probability density of sighting, a function of the density of search ϕ there, must equal a constant G or, if it cannot, ϕ must be zero. In the case of random sweeps, this leads to the equation

$$\left.\begin{array}{l} \phi = \ln(g/G) = \ln(2/AG) + \ln\left[1 - (q/A)\right] \\ \text{if } \phi \text{ is positive, otherwise } \phi = 0 \end{array}\right\} \quad (34)$$

But ϕ, the density of search, is the derivative of E with respect to q, and the whole equation can be written as a differential equation in terms of the variables defined in Eqs.(33)

$$\left.\begin{aligned} z'(x) &= \frac{dz}{dx} = k\left\{C - \ln(1+z)^2 + \ln\left[1 - \frac{x}{(1+z)^2}\right]\right\} \\ \text{where } k &= \frac{\alpha}{\beta} = \frac{\pi u^2 T_o^2}{WvT_o} \quad ; \quad C = \ln(2/\alpha G) = \phi(0) \end{aligned}\right\} \quad (35)$$

This equation may be integrated numerically, for different values of k and C, out to $x = x_m$, where z' goes to zero. The value of z at that point, z_m, times $\beta = WvT_o$, is equal to the total search effort E(T) expended; the value of x_m, times $\pi u^2 T_o^2 = \alpha$, is equal to the total area searched, $Q = \pi(r_m)^2$, and the value of z'(x) times 1/k is equal to the density of search ϕ at the radius $r = \sqrt{\alpha x/\pi}$. In other words

$$\left.\begin{aligned} &z'(x) \text{ goes to zero at } x = x_m, \text{ where } z = z_m \\ &\text{Total search effort } E_m = \beta z_m = WvT_o z_m \\ &\text{Total area searched } Q = \pi r_m^2 = \alpha x_m = \pi u^2 T_o^2 x_m \\ &\text{Search density at } r = \sqrt{\alpha x/\pi} < r_m, \text{ is} \\ &\qquad \phi = W/S = (1/k)z'(x) = (WvT_o/\pi u^2 T_o^2)(dz/dx) \\ &\text{Probability density of presence of target at time} \\ &\text{and place of search} \quad g(x) = \frac{2/\alpha}{(1+z)^2}\left[1 - \frac{x}{(1+z)^2}\right] \end{aligned}\right\} \quad (36)$$

The probability of success in the whole search is then

$$\mathcal{P} = \int_0^Q (1 - e^{-\phi})\, g\, dq = \alpha G \int_0^{x_m} (e^{\phi} - 1)dx = 2e^{-C} \int_0^{x_m} (e^{z'/k} - 1)dx \quad (37)$$

which also can be evaluated numerically.

A few examples of the results are shown in Figs.19 and 20. Two values of the parameter $k = \alpha/\beta = (\pi u T_o/W)(u/v)$ (?) were used, 1 and 2. Since α is the area of presence of the target at the instant the search starts and β is the area that could have been searched effectively ($\phi = 1$) in the delay time T_o, k = 1 represents cases where the search started before the target could get very

Fig.19. Curves for determination of searched area Q and probability of detection $\mathcal{P}$ as functions of total search effort E = WvT (?), for the retiring search spiral of Eq.(35).

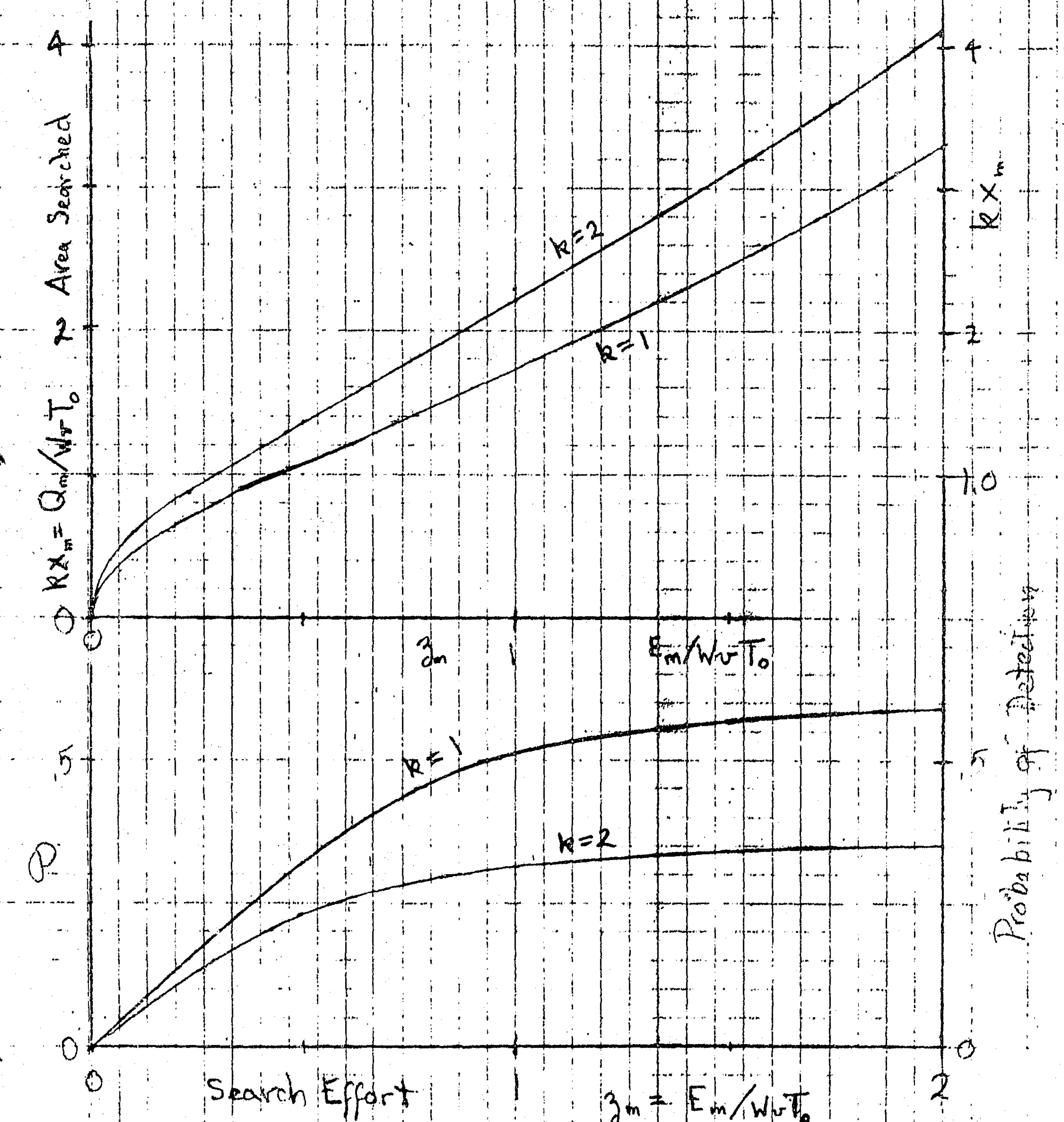
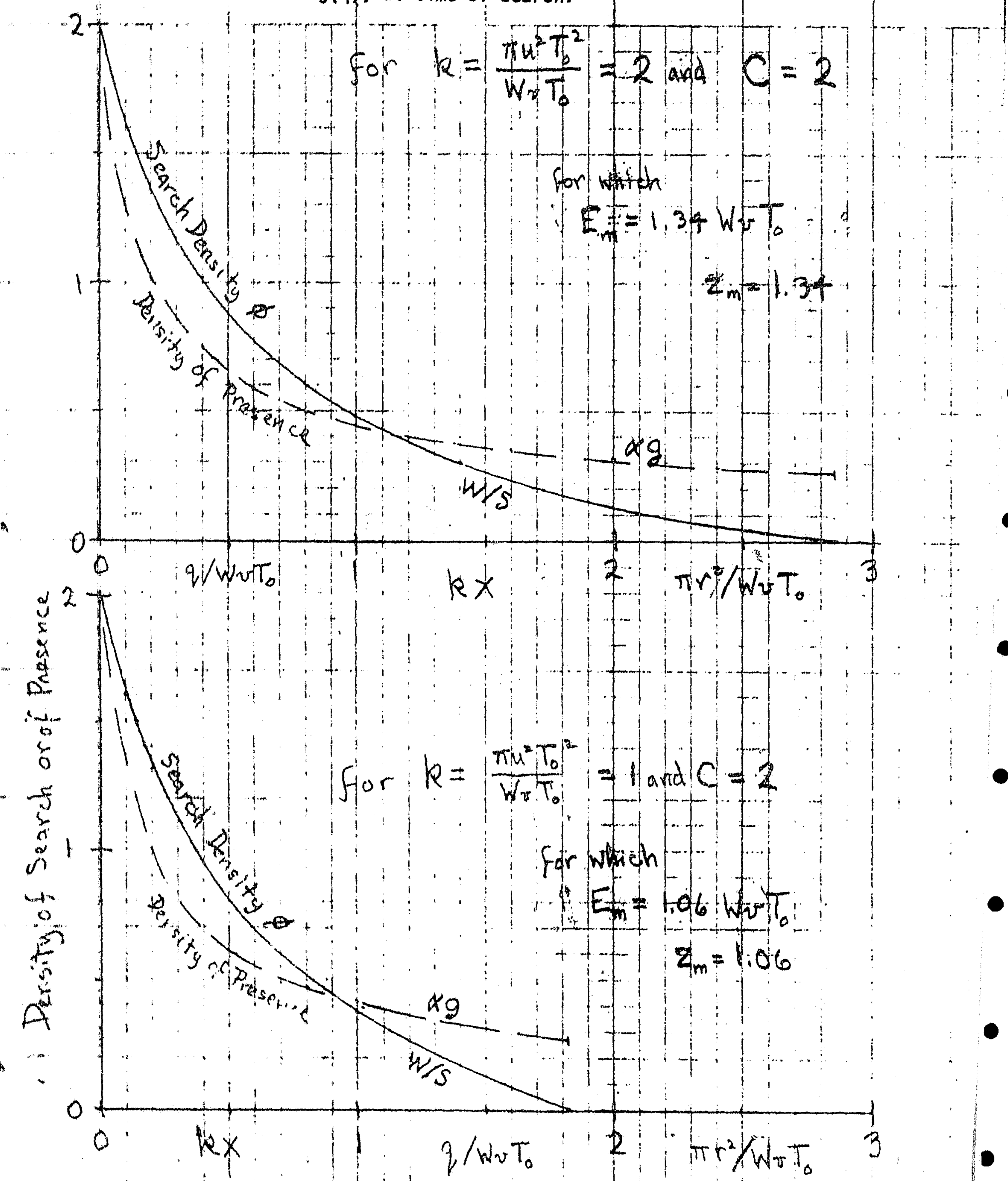


Fig. 20. Curves for search density ϕ as function of area q already covered, for two values of C and k. Dashed line is proportional to density of presence, g(q), at time of search.



far away from the origin. Case $k = 2$ represents searches where the target got more of a start, and the searcher had less chance of catching up.

We note that the probability of detection for $k = 1$ is nearly twice that for $k = 2$, and that neither probability increases much after E_m becomes larger than WvT_o, the area searched in time $t = T_o$. After covering the central region, the area covered, Q, rises slowly as search effort E is increased, but then begins to rise faster, as the quadratically increasing area of presence increases faster than the search can keep up. More area has to be covered for the $k = 2$ case than for $k = 1$, which means that it has to be covered less densely and thus the probability of detection suffers.

Figure 20 shows typical curves of search density ϕ for the same two value of k, but for specific values of C and thus of z_m. As expected, ϕ is greatest at the beginning of the search, near the origin, where the target was spotted. It has to cover a greater area for $k = 2$ than for $k = 1$ because the area of presence is always greater in the $k = 2$ case. The dashed lines are proportional to the probability of presence g(x) at the time the search reaches $x = \pi r^2/\alpha$. This quantity is not yet zero when the search ends, at $x = x_m$, because the area of presence extends beyond the area searched, as required by (23). Thus the results more or less bear out our preconceptions, though it is doubtful that intuition would have advised cutting off as early as $t = T_o$ or would have insisted on such a high concentration of search density close to the origin as is evidenced by Fig. 20.

Another approach to a related problem, using game theory, has been discussed by Danskin (1968).

6.6 Search of Discrete Sites.

There are operational situations that can be more easily modelled in terms of the search of discrete sites (boxes, in more mathematical jargon). The individual sites may themselves be separated areas, or have more complicated structure. All we need to know is the relationship between the effort expended in search at a site and the probability of discovery of the searched-for object. Here the "target" may not be a unique object that may be in one site or another but not in both; it may be in several sites simultaneously. So the probabilities of presence γ may not have to add up to unity. For example, the search may be to locate the failure in a complex piece of equipment; more than one failure may be present. An interesting example of this sort of search problem is the strategy of search for ores or oil. Other examples of equal complication are those connected with police search. All we can do in this Section is to report a few simple models, in the hope that they may be of more use than no model at all, or than a model too complex to ~~evaluate in practice~~ put to practical use.

From one point of view these discrete site-search problems are simpler than the area search problems we have been discussing earlier. We did not treat them first because the area-search problem is the classical search problem, dealt with first and, to date, of more practical utility.

6.61 An Analogue of Area Search.

The discrete analogue of the classical allocation of search effort, given in Eqs.(23) to (31) for the area case, is the following one:

There are N sites, the probability of presence of the target in the j'th site is γ_j, where $0 \leq \gamma_j \leq 1$ but $\sum \gamma_j$ is not necessarily unity. We rank-order the sites in a decreasing order of probability, so that $\gamma_j \geq \gamma_{j+1}$. The probability that a target is discovered in site j, if it is present, is related to a quantity we shall call the search effort ϕ_j in j by a function $P(\phi)$, that satisfies the specifications given in (5) and is the same function for every site. We wish to distribute the search effort $E = \sum \phi_j$ over the N sites so as to maximize the sum of the probabilities $\gamma_j P(\phi_j)$. See Charnes and Cooper (1958) for details.

This is actually a simpler problem than that of Eqs.(23) to (26) since in the present case the only parameters to evaluate are the γ's, instead of the γ's and α's of Eq.(24). The added complication may be achieved by assuming the search effort has different powers in different sites, so that the probability of detection in site j is $P(\alpha_j \phi_j)$ instead of $P(\phi_j)$. In this case the problem is completely parallel to that of Section 6.4. Because of the simplicity of the results and the wide range of applicability of the formulas, we will go into details only for the case of the exponential formula

$$P(u) = 1 - e^{-u}$$

Assuming equal searchability of each site (i.e., that $\alpha_j = 1$ for each site) our problem is to

$$\text{Maximize } \mathcal{P}(E) = \sum_{j=1}^{N} \gamma_j (1 - e^{-\phi_j})$$

subject to the requirement $\sum_{j=1}^{N} \phi_j = E$. The standard procedure is to minimize

$$J(E) = \sum_{j=1}^{N} \left[\gamma_j e^{-\phi_j} + \lambda \phi_j\right] \tag{38}$$

with parameter λ to be adjusted so that the sum of the ϕ's equals E. The process of solution is as follows:

1. Compute the sequence $\ln(1/\gamma_j)$, increasing with j. Also compute the partial sums $K_n = \sum_{j=1}^{n} \ln(1/\gamma_j)$ and the sequence $L_n = (1/n)K_n - \ln(1/\gamma_n)$, that also increases with n.

2. The minimization is accomplished by setting the derivative of J(E) with respect to ϕ_j equal to zero. Thus $\lambda = \gamma_j e^{-\phi_j}$. The requirement that $\sum \phi_j = E$ leads to the formula $\ln\lambda = -(1/n)(K_n+E)$ and thus to the elimination of λ from the formulas for ϕ_j and $\mathcal{P}_j$.

3. When $L_1 = 0 \leq E \leq L_2$ site 1 only is searched (the site with the largest value of probability of presence γ). Then

$$\phi_j = E \text{ for } j = 1 \text{ and } 0 \text{ for } j > 1 \quad \text{and } \mathcal{P} = \gamma_1(1 - e^{-E})$$

When $L_n \leq E \leq L_{n+1}$ only the most probable n sites are searched, and

$$\phi_j = (1/n)(K_n + E) - \ln(1/\gamma_j) \text{ if } j \leq n, \quad = 0 \text{ if } j > n$$
$$\mathcal{P} = \sum_{j=1}^{n} \mathcal{P}_j \; ; \; \mathcal{P}_j = \gamma_j - \exp[-(1/n)(K_n+E)] \text{ if } j \leq n \qquad (39)$$

When $L_N < E$ all N sites are searched and

$$\phi_j = (1/N)(K_N+E) - \ln(1/\gamma_j) \; ; \mathcal{P}_j = \gamma_j - \exp[-(1/N)(K_N+E)]$$
$$E = \sum_{j=1}^{N} \phi_j \quad ; \quad \mathcal{P} = \sum_{j=1}^{N} \mathcal{P}_j$$

insert here

If, after expending effort E, the target is not yet found, and it is decided to spend an extra $\Delta E = E' - E$, recompute (39) using E' instead of E and add search efforts $\phi_j' - \phi_j$ to each site j.

An example of this solution is shown in Fig.21 for four sites. For four or fewer sites the calculations can be made by nomogram if very approximate solutions are good enough. The nomogram is shown in Fig.22. To use it a line is drawn from γ_1 (column 1) to γ_2 (column 2) on the right-hand side, to locate point u on the B vertical; lines from u to γ_3 to locate v on the C vertical and from v to γ_4 to locate w on the D column. To see how many sites are to be searched we locate, on the central column, the intersections of lines drawn from γ_1 on the right to E on the left column marked 1, from u to E on the left side column marked B, from v to E on column C and from w to E on column D. From the w to E of D intersection on the central scale we draw a line to

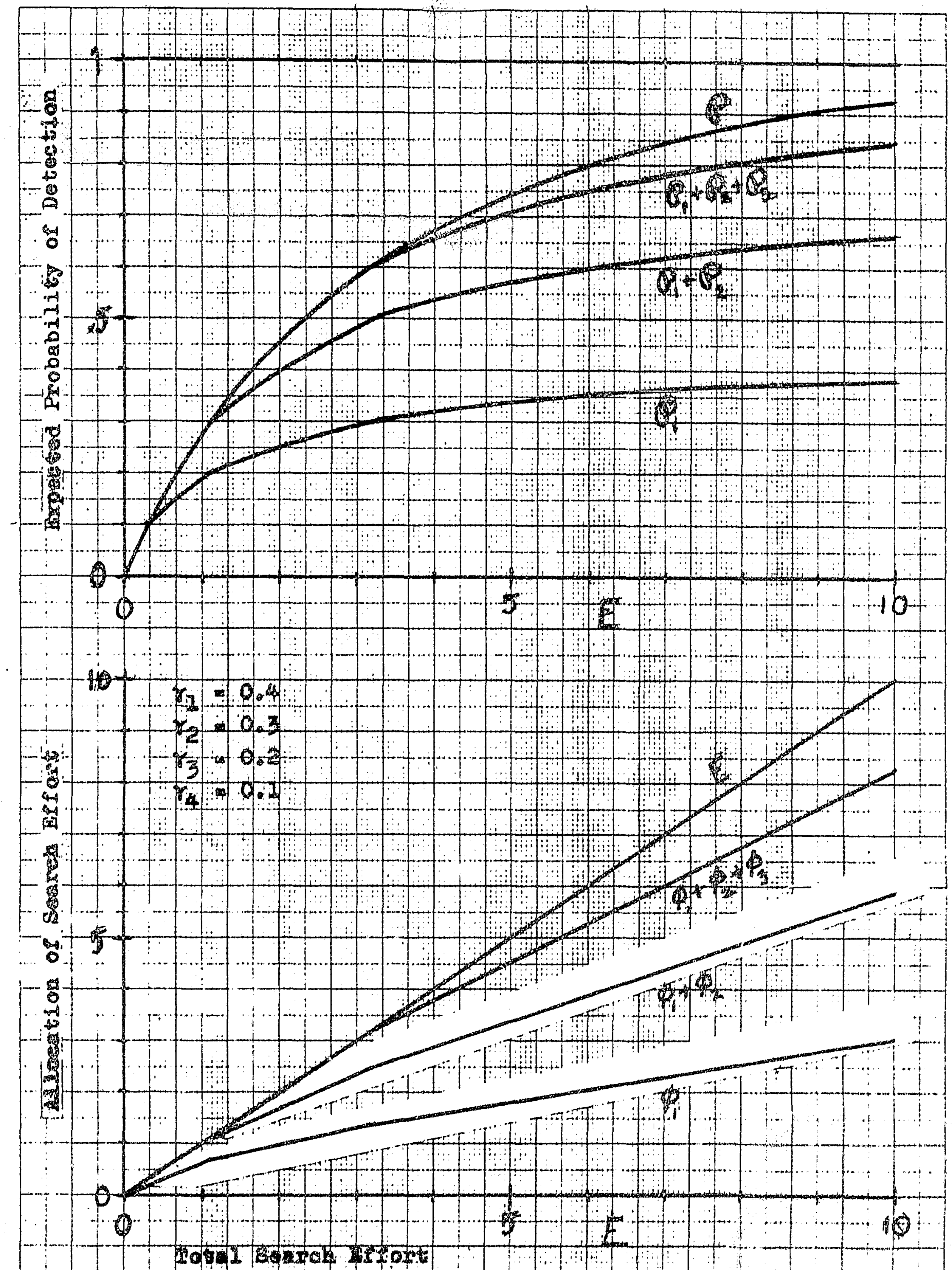


Fig.21. Typical allocation of search effort among discrete sites.

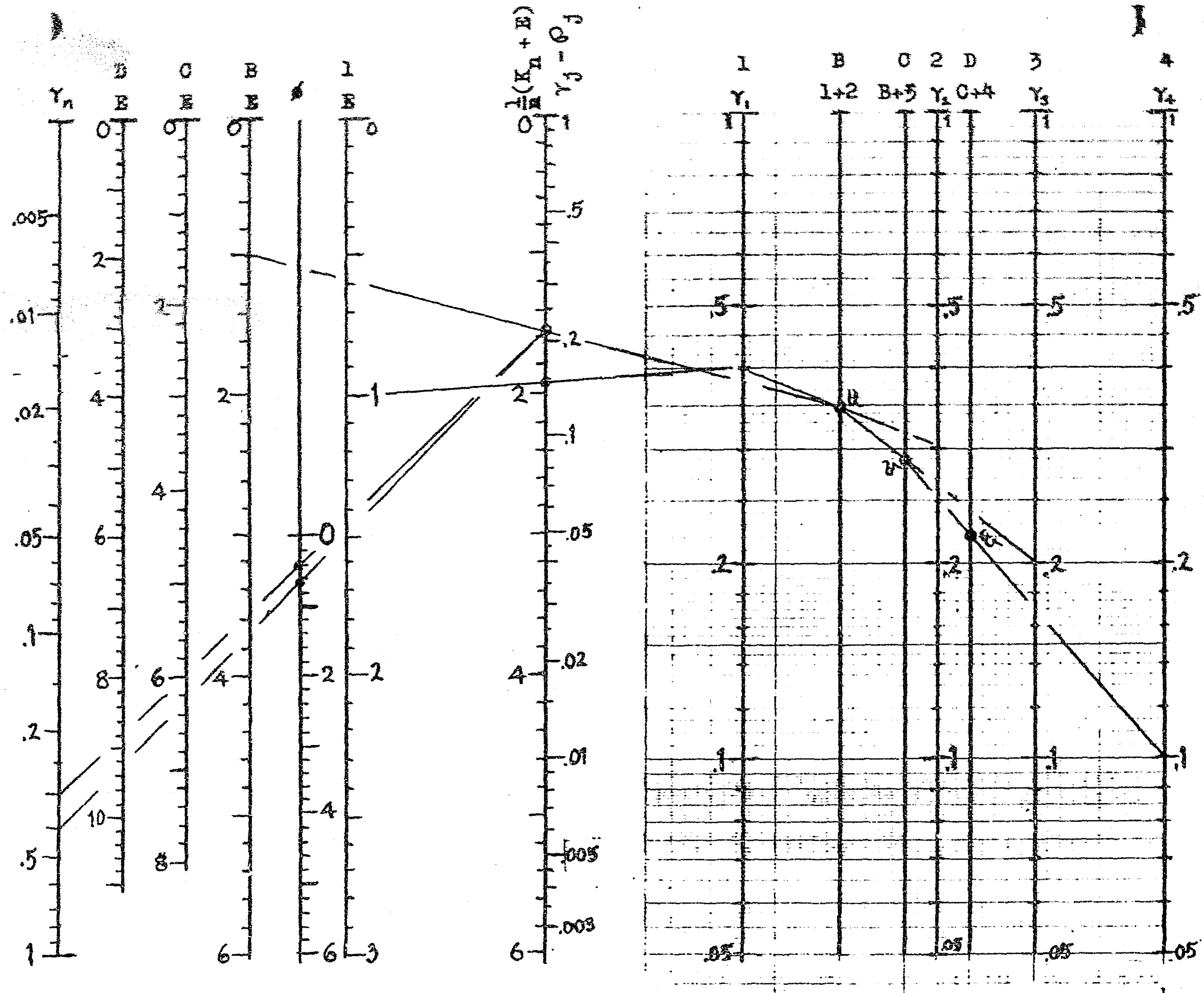


Fig. 22. Nomogram for search allocation

γ_4 on the left-most column marked γ_n. If this line intersects the lower, calibrated half of the scale marked ϕ then all sites are to be searched. If it cuts above the $\phi = 0$ on this scale then site 4 is not to be searched. Then take the v to E of C intersection with the central scale and see whether a line from this intersection to γ_3 on the γ_n column comes below the 0 mark on the ϕ column, and so on until an intersection below the 0 is obtained.

The example illustrated is the case of E = 1 of Fig. 21, for $\gamma_1 = 0.4$, $\gamma_3 = 0.3$, $\gamma_2 = 0.2$ and $\gamma_4 = 0.1$. At E = 1 only the first two sites have intersections below the zero of the ϕ scale, so we use the intersection of the u to E of B line on the central scale, corresponding to $\gamma_j - \mathcal{P}_j = 0.22$. The line from this point to $\gamma_1 = 0.4$ on the leftmost scale intersects the ϕ scale at $\phi_1 = 0.6$ and the line from this point to $\gamma_2 = 0.3$ gives $\phi_2 = 0.4$; these are the two search efforts in the sites searched. The probabilities of success are obtained from the value of $\gamma_j - \mathcal{P}_j = 0.22$; $\mathcal{P}_1 = 0.4 - 0.22 = 0.18$, $\mathcal{P}_2 = 0.3 - 0.22 = 0.08$ and thus $\mathcal{P} = 0.26$. Once learned, the procedure is straightforward and fairly rapid, though the results have barely 2-significant-figure accuracy.

Solutions for other kinds of discrete search problems have been developed by Gluss (1959) for the allocation of effort in testing for failures in a complex electronic system. In this case, instead of a probability of detection depending on a continuous effort function, times are assumed for checking out each site and probabilities are given that the specified times will find the error. A dynamic programming technique is developed to determine the order in which the sites are to be searched so that expected total time is minimized.

6.62 Detection Errors.

We have noted, at several points during our discussion of continuous search coverage, the complexities that arise when false targets are present. Indeed, in most cases when laying out search strategy in practice, we are forced, at present, to assume that the presence of false targets will not not alter the structure of the theoretical models, aside from reducing the magnitude of some of the parameters. As we have said, this is one of the reasons that these parameters (sweep width, observer's mean velocity, etc.) should be measured under operational conditions, rather than taking values from laboratory measurements.

In the case of the search of discrete sites the inclusion of false targets (or detection errors) again adds complexity to the analysis. In many of these cases the complexities are as great a hindrance to the practical use of the theoretical results as they are with the continuous search results. In a few cases, however, one can simplify the assumptions sufficiently to produce usable solutions. These solutions may at least indicate the structure of yet another search operation, that of searching for oil or for ore. Here the employment of a certain amount of search effort at a site may produce a positive or a negative indication. If the indication is positive, it is still possible that the desired oil or ore is

nevertheless absent from the site. Likewise there is a non-zero chance that a negative indication may be erroneous. A detailed study of the observer's process of deciding whether he has found the target, and how this affects the cost of both kinds of error, has been discussed by Pollock (1964) and others (see Pollock, 1971, for a bibliography). We need only take the results here, to show how they modify the allocation of search effort. The example we use to illustrate our formulas is a simplified model of prospecting for oil or ore.

In looking for new sources of minerals one first looks for possible sites for more detailed study, by searching for particular geological formations or other characteristics that have been present in previous successful strikes -- including simple proximity to known sources. This preliminary exploration, partly in the field and partly from maps, yields estimates of the likelihood of striking "pay dirt" at a number of possible sites. From this list of a priori probabilities of presence of ore, one must lay out a strategy for the more expensive part of the prospecting operation. Of course one could blindly sink all further effort into the single highest rated site, but it might be better in the long run to utilize these a priori probabilities as fully as possible in deciding whether and how to go ahead. These estimates of the probability of presence of the mineral (which we can again call γ_j) at each possible site j will be changed as the operation progresses, but at the beginning, when the first plans are made, they are the only measures available.

At each likely site a survey must be made, using sonic or gravitational or magnetic or electrical equipment, to sharpen our estimate of the probability of presence of the mineral. The preliminary plan must decide how extensive such a survey should be. Again, the preliminary estimates of survey effort may be modified later, but the initial allocation of effort must be made on the basis of the a priori probabilities γ. By the end of the survey a decision must be made, whether to abandon further effort at that site or to commence excavation (or drilling), hopefully to obtain actual samples of the desired mineral.

The excavation or drilling is usually much more expensive than the instrumental survey, and one hopes that the survey has reduced the chance of an erroneous decision to excavate, with no ore to show for the digging. (An alternative analysis of this decision process is given by MacQueen and Miller, 1960).

It may be that the result of the survey measurements is to increase the chance of deciding to excavate. With no survey we may be disinclined to dig or drill; with a very extensive survey

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we would have reached a fairly precise estimate of the worth of excavation. At the time of the preliminary plans our best [redacted] guess as to the likelihood of a decision to excavate at site j would be the *a priori* probability γ_j. Thus one possible forecast of the results of the instrumental survey is that, as the survey effort is increased from zero to some large value, the chance of our deciding to investigate would start from zero and approach the value γ_j asymptotically. In other words this chance would depend on the effort ϕ_j expended on the survey something like the function $\gamma_j(1-e^{-\phi_j})$, with ϕ_j being proportional to the expenditure involved in making the survey at site j.

Unless the instruments used in the survey are perfect, a certain fraction of times the decision is made to excavate, it will have been a wrong decision and a lot more money would have been needlessly spent. The crucial question is; how does the fraction of "dry holes" to "strikes" depend on the amount of effort spent on the instrumental survey? Decision theory does not give us an unequivocal answer to this question. Indeed the answer depends on the nature of the equipment used in the survey and on *how* it is used. All we can do here is to make a few not unreasonable guesses as to possibilities and work out models to correspond. In the end the choice of model and the values of its parameters will have to be decided on the basis of operational experiments, just as was done for the model of search for a submarine by a plane. In view of the costs of mineral prospecting and the value of a "strike", such a series of measurements would seem to be a worth while investment.

At one end of the sequence of possibilities is to assume that the amount of effort expended on the instrumental survey changes

the probability of reaching a decision to excavate but does not alter the ratio between success and failure, if excavation is carried out. Put in terms of expected monetary costs and returns, this limiting model is:

> *A priori* probability of presence of ore at site j, reached from the preliminary exploration, is γ_j, the only quantitative estimate available at the time of the initial planning.
>
> Estimated returns from site j, if ore is present and discovered by excavation, is R_j.
>
> Expected cost of excavation to "prove out" site j is D_j.
>
> Expected cost of instrumental survey at j to help decide whether to excavate is C_j.
>
> *A priori* probability that a decision will be made to excavate at j is $\gamma_j(1-e^{-\beta C_j})$.
>
> *A priori* probability that this decision will be correct, if made, is assumed in this model to be equal to γ_j, the *a priori* probability of presence of the ore. Until the instrumental survey is made, we have no other information beside γ_j and the estimated costs; we must use them in laying out our preliminary strategy.

Thus the expected return from site j, if survey effort costing C_j were to be expended there, is

$$Q_j = \gamma_j(\gamma_j R_j - D_j)(1 - e^{-\beta C_j})$$

In other words the expected net return from site j, if excavation is decided, is $S_j = \gamma_j R_j - D_j$; there is a chance γ_j that the excavation succeeds, returning R_j, but a certainty that the excavation will cost an expected D_j. Thus the variational problem representing this case is:

$$\left.\begin{array}{l}\text{Maximize } Q = \sum_j\left[\gamma_j S_j(1-e^{-\beta C_j}) - C_j\right] \\ \text{Subject to the requirement that } \sum C_j = C\end{array}\right\} \quad (40)$$

This is quite similar to the Eqs. (38) for simple search of N sites, with βC_j substituted for ϕ_j and $S_j\gamma_j$ for γ_j. The solution also is quite similar;

Rank the sites in decreasing order of magnitude of $\gamma_j S_j$, then calculate the sequences $K_n = \sum_{j=1}^{n} \ln(\beta\gamma_j S_j)$ and $L_n = K_n - n\ln(\beta\gamma_n S_n)$.

When $L_1 = 0 < C < L_2$ only site 1 (with the largest γS) is to be considered. The survey at site 1 is planned to cost C and the expected return is

$$Q(C) = \gamma_1 S_1(1 - e^{-\beta C})$$

When $L_n < C < L_{n+1}$ (assume that $L_{N+1} \to \infty$) then only the n most promising sites are surveyed, the survey cost allocation for site j being

$$C_j = \frac{1}{\beta}\left[\ln(\beta\gamma_j S_j) - \frac{1}{n}K_n\right] + \frac{1}{n}C \quad (j \le n)$$

and the expected return is

$$Q(C) = \sum_{j=1}^{n}\gamma_j S_j - \frac{n}{\beta}\exp\left[\frac{1}{n}(K_n - \beta C)\right] - C \qquad (41)$$

We can now adjust the survey cost C to produce the greatest return $Q(C)$, by setting the differential of Q with respect to C equal to zero. We find the value of n for which

$$Q_{max}(n) = \sum_{j=1}^{n}\gamma_j S_j - \frac{1}{\beta}(K_n + n) \quad \text{is greatest}$$

The corresponding optimal survey cost allocation is then

$$C_{max}(n) = K_n/\beta \quad \text{and} \quad C_j = (1/\beta)\ln(\beta\gamma_j S_j) \quad (1 \le j \le n) \qquad (42)$$

The optimal value of C is for $n = N$ (all sites surveyed) unless $\beta\gamma_j S_j$ is less than unity for some site; in which case n is the largest value of j for which $\beta\gamma_j S_j > 1$.

To assume, as this model does, that the effect of the instrumental survey will have no effect on the ratio between success or failure of the excavation, but only on the chance of of deciding to excavate, is perhaps the most pessimistic assumption to make. We see that it results in the rule that the sites with the largest values of $\gamma_j S_j$ should be surveyed first.

At an opposite extreme is the assumption that the probability of making a decision to excavate at site j is independent of the amount of survey effort put in at site j (i.e., it has the value γ_j for all values of C_j) but the probability that the excavation is successful increases, from γ_j for $C_j = 0$, asymptotically to unity as $C_j \to \infty$. In other words the expected probability of deciding to excavate site j is γ_j and the expected cost of such excavation is $\gamma_j D_j$, but the expected return from a possible successful excavation is $\gamma_j\left[1 - (1-\gamma_j)e^{-\alpha C_j}\right]R_j$, increasing as C_j is increased. Each of the N sites chosen may be excavated; the detailed instrumental survey is used to improve the chance that the excavation is successful. This effect would be greater when γ_j is near 1/2 than when γ_j is near unity (when a detailed survey may not be needed).

To be realistic, this model must represent a separation of the decision process into three steps instead of two (see Engel, 1957, for another such model):

1. On the basis of map and field exploration a list of sites is chosen that have a good chance of containing the searched-for mineral.
2. Then an initial survey, using simple equipment, costing C_0, is run on all chosen sites. On the basis of this survey probabilities γ_j are assigned. It may be that some γ's are near enough unity to justify going ahead with the excavation. Also those sites turning out to have γ's less than some lower limit (presumably 1/2 or even greater) will be discarded from the list.
3. On the basis of the solution of the variational problem, given below, some or all of the sites remaining may have further, more intensive surveys applied before the decision to excavate is made.

The variational problem is thus:

$$\left.\begin{aligned} &\text{Maximize } Q = \sum \gamma_j \left[1 - (1-\gamma_j)e^{-\alpha C_j}\right] R_j - C_j - \gamma_j D_j \\ &\text{subject to the requirement that } \sum C_j = C \end{aligned}\right\} \quad (43)$$

and the procedure for solution is;

Rank the sites in descending order of $\gamma(1-\gamma)R$, with $\gamma_1(1-\gamma_1)R_1$ being the largest (if the R's are equal and if all the γ's are greater than 1/2, the sites will be in <u>increasing</u> order of the γ's, the <u>smallest</u> γ being first).

Calculate the ascending sequences $K_n = \sum_{j=1}^{n} \ln\left[\alpha\gamma_j(1-\gamma_j)R_j\right]$ and $L_n = K_n - n \ln\left[\alpha\gamma_n(1-\gamma_n)R_n\right]$.

When $L_1 = 0 < C < L_2$, make the intensive survey only in site 1, that has the greatest uncertainty $\gamma_1(1-\gamma_1)R_1$ in expected return; decide on excavating the other sites on the basis of the γ's obtained from the initial survey (using some decision procedure that results in a probability γ_j of deciding to excavate site j).

The expected payoff is then

$$Q(C) = Q(0) + \gamma_1(1-\gamma_1)R_1(1-e^{-\alpha C})$$
$$Q(0) = \sum \gamma_j(\gamma_j R_j - D_j) - C_o$$

When $L_n < C < L_{n+1}$ (assume that $L_N \to \infty$) only the first n sites are given a more detailed survey, the j'th costing C_j; the rest are decided on the basis of the initial survey. The recommended costs C_j of the further survey, and the expected payoff of the plan are then

$$\left.\begin{aligned} C_j &= \frac{1}{\alpha}\left\{\ln\left[\alpha\gamma_j(1-\gamma_j)R_j\right] - \frac{1}{n}K_n\right\} + \frac{1}{n}C \quad \text{for } j \le n \\ Q(C) &= Q(0) + \sum_{j=1}^{n}\left\{\gamma_j(1-\gamma_j)R_j - \frac{1}{\alpha}\exp\left[\frac{1}{n}K_n - \frac{\alpha}{n}C\right]\right\} - C \\ Q(0) &= \sum_{j=1}^{n}\gamma_j(\gamma_j R_j - D_j) - C_o \end{aligned}\right\} \quad (44)$$

As before, we can now determine the cost of the detailed survey allocation that will produce the greatest expected return. The result is, for the largest value of n for which $\alpha\gamma_n(1-\gamma_n)R_n$ is greater than unity,

$$\left.\begin{aligned} C_{max}(n) &= (K_n/\alpha) \quad \text{and} \quad C_j = \frac{1}{\alpha}\ln\left[\alpha\gamma_j(1-\gamma_j)R_j\right] \\ Q_{max}(n) &= \sum_{j=1}^{n}\gamma_j(\gamma_j R_j - D_j) - \frac{1}{\alpha}(K_n + n) - C_o \end{aligned}\right\} \quad (45)$$

An example of this solution is shown in Fig. 23.

Thus the final results of these two alternative models are similar in some respects and contrasting in others. They both point to the importance of determining the value of the parameter β or α, measuring the effect of the cost C_j of the instrumental survey on the improvement of the probability that excavation will be successful. Once even a crude value of this parameter can be obtained, one or the other of the models (or another, perhaps, intermediate between the two) can be used to estimate, at the start, the probable worth of the campaign and an initial estimate of the allocation of effort that may be involved. These estimates will be altered as the search goes on, but at the beginning, the appropriate model, with the best estimates of the values of the γ's, R's, D's and of α or β, is the only way it will be possible to estimate, quantitatively, the projected campaign.

For example, both results show that a site j for which the a priori estimate of $\beta\gamma_j S_j$ or $\alpha\gamma_j(1-\gamma_j)R_j$ is less than unity is probably not worth including in the campaign. It is better to reduce the number of sites to those that cannall be covered by some amount of instrumental survey. The particular model that has been shown to be appropriate will then indicate which sites deserve more coverage than others.

As a final comment, the models discussed here may be useful in other than prospecting operations. For example, the process of testing a few samples from each manufactured production lot would be the preliminary exploration, determining γ; the more detailed "instrumental survey" might be the sampling of a larger fraction of the output. "Excavation" would be the testing and repairing or discarding each unit. Decision "not to excavate" would correspond to deciding to scrap the batch without further testing. Analogues in police investigation also come to mind.

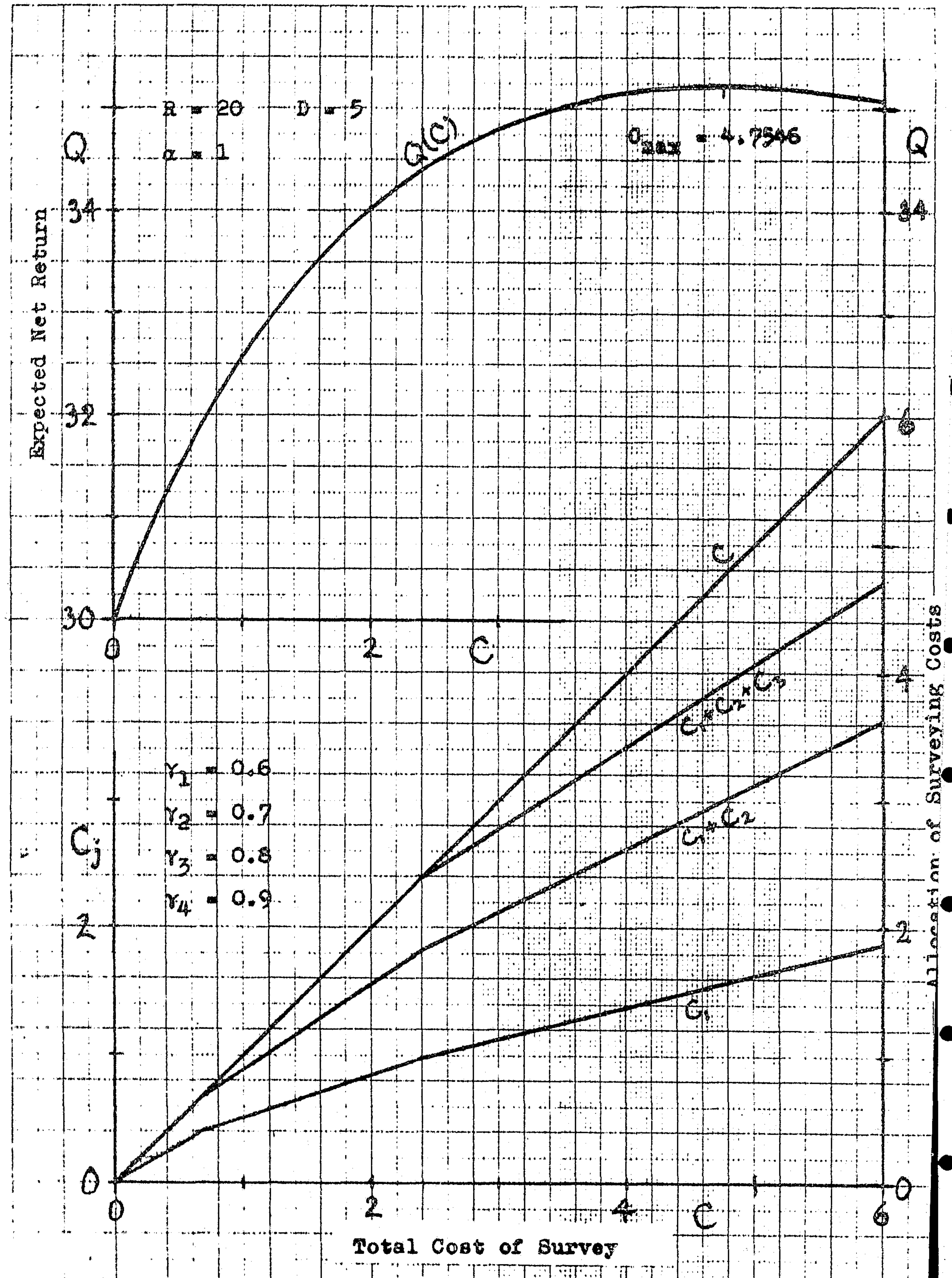

Fig.23. Solution of Eqs.(44) and (45) for 4 prospecting sites.

6.7 Search for an Active Evader.

Heretofore we have concentrated on the search for an object that stays put, or moves without relation to the details of the search path. A still more difficult task is the devising of search strategies for a target that is aware of the search and tries to evade. In fact there are very few such solutions that are realistic enough to be of practical use. All of them, to date, utilize the Theory of Games (see Chapter 5), a theory not too productive of useful results as yet. Still, some of the results may qualitatively indicate the desired strategy.

As seems to have been the usual case in search theory, a continuous, rather than a discrete example was first worked out. The example (see Morse and Kimball, 1946, page 105) is an over-simplified model of an air patrol to prevent a submarine from getting through a long strait of varying width, with the sub-marine able to submerge part but not all the time, and the cross-over barrier patrol having varying degrees of coverage. An alternative example is that of a patrol to prevent infiltration across a length L of the border of a country.

As with many game theory solutions, the strategies of both sides, the infiltrators (I) and the repulsive patrol (R), must vary their actions along the border; otherwise the opposition will learn these actions and devise means of circumventing them. Only by continuously varying actions, according to a prescribed probability distribution, can the opponent be kept guessing. Suppose side I sends each infiltrator at random across the border with a probability density $\Psi(x)$ that he cross at point x (so that $\int_0^L \Psi(x)\,dx = 1$). And suppose that side R places its patrol at random along x with a probability density $\phi(x)$ that the patrol covers x (here also $\int_0^L \phi(x)\,dx = 1$). And, finally, suppose that if a patrol happens to be covering x and an infiltrator happens to try crossing at x then, the probability that he will be prevented from crossing is P(x). This probability will vary with x, depending on the terrain; in heavily wooded or mountainous country P would be small, for example. The expected fraction of infiltrators that are prevented from crossing the border will, in the long run, be

$$J = \int_0^L \phi(x)\,\Psi(x)\,P(x)\,dx \qquad (46)$$

The problem for side I is to adjust the likelihood of crossing Ψ so that J is as small as possible; that for side R is to adjust the frequency of patrols ϕ so that J is as large as possible. To be safe, side I should arrange Ψ so that no action by R can make J larger, and side R should arrange ϕ so that no action by I can make J any smaller.

Taking side R first, note that, in integral J, if the product ϕP is smaller, for some range of x, than it is elsewhere, then if side I finds this out, more infiltrators will be sent through the "weak" range of x and J will be reduced in value. Therefore the safe strategy for R is to make ϕ inversely proportional to P(x) (heavy patrolling where P is small, light patrolling where P is large). To be more precise, side R should make

$$\left.\begin{array}{l} \phi(x) = [1/N(L)P(x)] \;\; ; \;\; N(L) = \int_0^L [1/P(x)]\,dx \\ \text{in which case the fraction of infiltrators} \\ \quad \text{prevented from crossing is} \\ J(L) = [1/N(L)] \int_0^L \Psi(x)\,dx = [1/N(L)] \end{array}\right\} \qquad (47)$$

no matter what I does about the shape of $\Psi(x)$.

However unless side I does the same thing with Ψ, side R could modify ϕ so as to increase J. For example $\Psi(x)$ might be $[h/P(x)]$ over a smaller range L_h of L, for which $P(x) < H$ and be zero when $P(x) > H$. In this case

$$\Psi(x) = \begin{cases} [1/N(L_h)P(x)] & \text{over } L_h \\ 0 \text{ over the rest of } L \end{cases} \qquad (48)$$

where $N(L_h) = \int_0^{L_h} (1/P)\,dx$. If side R stuck to the patrol density ϕ of Eq.(47), the value of J would still be $[1/N(L)]$, but if R learned of the change to the Ψ of Eq.(48), he can change ϕ to increase J. For example he can make ϕ equal to the Ψ of Eq.(48), omitting any patrolling along $L - L_h$, where there is no infiltration. In that case $J(L_h)$ would equal $[1/N(L_h)]$, which is larger than $[1/N(L)]$ because L_h is smaller than L. Of course, if R continued to use this patrol density and side I learned of it, he could send infiltrators through the unpatrolled length $L - L_h$ without any loss.

Therefore the safe strategy for both sides is to have both Ψ and ϕ equal the $\phi(x)$ of Eq.(47).

Finally, there should be mentioned the discrete cases involving a search for a conscious evader. The problem of a number of discrete sites, where the evader can hide and the searcher may look, seems to be a very difficult one to solve. A start at a solution has been made by Norris (1962) for the very simplified case where the search is conducted in a series of discrete "looks" into the different sites, with specified probabilities q_j of discovering the evader if he is in site j when that site is looked into. As with other game theory solutions, this requires a mixed strategy solution, with the evader moving from site to site, between looks of the searcher, with specified probabilities that he make the change, plus specified probabilities for being initially in the different sites. The searcher must also use a mixture of strategies, each of which consists of a series of looks at a specified sequence of sites.

The game is determined by alloting quanta of gains to the evader every time the searcher looks but does not find him and costs every time he changes sites. Norris solved the case for two sites in some completeness. He found that if the cost of changing sites is larger than some limit, the best strategy for the evader is to choose a site initially, with a probability P of going to site 1 and 1 - P of going to site 2, and then staying put. These probabilities are determined by the relative magnitudes of the probabilities q_1 and q_2 of being discovered (which are presumed known to both sides). If the searcher assumes that the evader has hidden according to these safest probabilities, he then can look in one of the sites, thus changing the a priori probability P into an a posteriori probability (if his look does not find the evader). (See Pollock, 1960, for further discussion). The desired sequences of looks are those which tend to keep these a posteriori probabilities oscillating within limits. The searcher must also use a mixture of these "good" sequences.

The same considerations also enter into the game when the evader can move from one site to another between looks. When more than two sites are involved the problem is considerably more complicated. Other aspects are treated by Neuts (1963).

Indeed, this part of the theory is not yet in shape to be useful in any real world situation. In fact, as we said in the beginning, search theory, in regard to practical applications, is still in an embrionic state.

6.8 Applications.

As indicated several times in this Chapter, although much of the basic structure of search strategy has been elucidated in the literature, the specific solutions appropriate to a given application are, for the most part, yet to be worked out. The search process enters into a surprisingly large number of our individual, as well as group, actions. We look for a book in the library or an item in a catalogue. Searches are conducted for a lost child, a fugitive, a buried city or pocket of oil, an enemy submarine or infiltrating division, a faulty component in an ailing piece of equipment or an error in a manufacturing process. Each of these searches has its own physical, procedural and economic boundary conditions; each requires considerable study and experimentation before a workable search strategy can be devised for it. In only a few cases have they been studied, measured and analyzed in detail.

To date, most of the practice of search theory has been in the military field (see, for example, Koopman, 1946, and the bibliographies of Enslow, 1966, and Dobbie, 1968). Much of the detail of these applications is, of course, buried in secrecy. The nature of the search for a person lost in a wilderness, and a few applications of the theory have been reported by Kelley (1973). Some applications in the search for flaws in equipment have been reported (see Gluss, 1959, for example) and a few reports in the field of prospecting (see Engel, for example). A small amount of work has been reported (Larson, 1972) on the police search problem, particularly in regard to the allocation of patrol effort. Practical applications in many other fields are still lacking.

As for the development of theory, some interesting progress has recently been made in the analysis of the effects of false targets on search strategy (see Stone, 1972 and Dobbie, 1973) and a little progress has been made into the immensely difficult problem of the search for a conscious evader. In general, however, one has the impression that the theory needs to be proved out by application in many more fields before [redacted] further mathematical superstructure is added. The subject is already topheavy enough.

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